Year 12 Problem-Solving Course

Section 4a: Combinatorics - counting

Exercise - Sketch Solutions

1. 1 dot in 3 boxes $= \binom{3}{1} = 3$

   2 dots in 3 boxes $= \binom{3}{2} = 3$

2. *i*) The 8 corners each have 3 faces painted.
   ii) The 12 sides that are not corners each have 2 faces painted.
   iii) The 6 cubes in the middle of a face have 1 face painted.
   iv) The 1 cube in the middle has no faces painted.

3. There are 4 possibilities for the first number and 3 for the second, giving $4 \times 3 = 12$ possibilities overall.

   2 dots in 4 boxes $= \binom{4}{2} = 6$

4. Let the number of year 12 girls be x and the number of year 13 boys be y

	Y12	*Y13*	*Overall*
Boy	$37 - y$	y	37
Girl	x	$43 - x$	43
Overall	45	35	80

$37 - y + x = 45$

$x - y = 8$

There are 8 more year 12 girls then year 13 boys.

[You don't actually need two letters/variables to solve this problem:

The only information you are explicitly given is

	Year 12	Year 13	Overall
Boy			37
Girl			43
Overall	45	35	

As we don't know the number of girls in Year 12 or boys in Year 13, we are going to have to use a letter/variable for one of them, so we may as well let the number of girls in Year 12 be x, just as was done above, and then, rather than introducing a new letter for the number of girls in Year 13, knowing that the total number of girls is 43 (i.e. using the second row) allows us to write $43 - x$ for the number of girls in Year 13 (again, as before):

	Year 12	Year 13	Overall
Boy			37
Girl	x	$43 - x$	43
Overall	45	35	

Then, rather than introducing a new letter for the number of boys in Year 13, knowing that the total number of Year 13 students is 35 (i.e. using the second column) allows to write $x - 8$ for the number of boys in Year 13 (so that $43 - x + x - 8 = 35$):

	Year 12	Year 13	Overall
Boy		$x - 8$	37
Girl	x	$43 - x$	43
Overall	45	35	

That there are 8 more Year 12 girls than Year 13 boys is then immediately obvious, without us even bothering about the number of boys in Year 12!]

5. If Hongeria scored 0 at half time, there is 1 possible score.
   If Hongeria scored 1 at half time, there are 2 possible scores.
   This continues until Hongeria scored 9 with 10 possible scores.
   After this, if Hongeria scored 10 there are 10 possible scores, etc.
   $1+2+...+9+10=55$
   $55+10\times10=155$

   For the second question, there are 28 goals and Alduras score 19 of them (and Hongeria 9 of them). So the number of ways of the scoreline reaching 19-9 comes from the number of ways of choosing which of the 28 goals Alduras score, i.e. ${}^{28}C_{19}$ (or from the number of ways of choosing which of the 28 goals Hongeria score, i.e. ${}^{28}C_{9}$, but both answers are correct since ${}^{n}C_{r} = {}^{n}C_{n-r}$).

6. After 1 sock, the worst situation is either 1B or 1R
   After 2 socks, the worst situation is 1B and 1R

After 3 socks, the worst situation is either 2B and 1R or 2R and 1B – there is always a pair.
For the second question, the only way to not have a sock of each colour is to keep picking socks of the same colour. There are more blue socks than red. The most unlucky could be is that if after 8 socks he has picked all of the blue ones. Then he HAS to pick a red. So the answer is 9 socks.

7. *a*) $\binom{8}{1} = \frac{8!}{7!1!} = \frac{8}{1} = 8$

$$\binom{8}{2} = \frac{8!}{6!2!} = \frac{8\times 7}{2} = 28$$

$$\binom{8}{3} = \frac{8!}{5!3!} = \frac{8\times 7\times 6}{6} = 56$$

$$\binom{8}{4} = \frac{8!}{4!4!} = \frac{8\times 7\times 6\times 5}{24} = 7\times 2\times 5 = 70$$

$$\binom{20}{5} = \frac{20!}{15!5!} = \frac{20\times 19\times 18\times 17\times 16}{120} = 19\times 17\times 16\times 3 = 15504$$

b) $\binom{8}{7} = \binom{8}{1} = 8$

$$\binom{8}{6} = \binom{8}{2} = 28$$

$$\binom{8}{5} = \binom{8}{3} = 56$$

$$\binom{8}{4} = \binom{8}{4} = 70$$

$$\binom{8}{6} = \binom{8}{2} = 28$$

$$\binom{8}{5} = \binom{8}{3} = 56$$

$$\binom{20}{15} = \binom{20}{5} = 15504$$

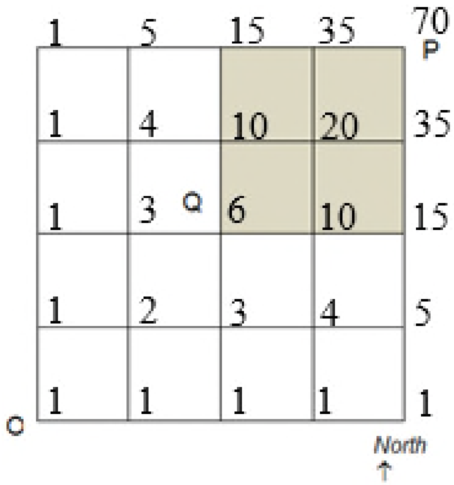
8. $a)\ \binom{18}{14} = \frac{18!}{14!4!} = 3060$

$b)\ \binom{38}{32} = \frac{38!}{32!6!} = 2760681$

$c)\ \binom{18}{14} \times \binom{20}{18} = 3060 \times 190 = 581400$

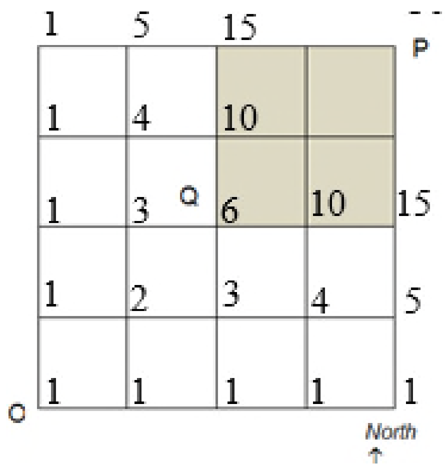
$d)\ \binom{66}{30}$

9.



In the above diagram, the number at each intersection of lines is the number of ways from O to that point. To get to each point you can only reach it from the point immediately below or the point immediately to the left. Hence each number is just the sum of the number below and the number to the left. (NB. This should resemble a certain triangle you use in your Core 1 Maths A Level!).

a) 70 ways
b) 6x6=36 ways. (Q to P is the same problem as O to P. 6 different ways from O to P and, for each of these, there are 6 different ways from Q to P.)



c) 15+15=30 ways