Year 12 Problem-Solving Course

Section 5: Geometry – angles, triangles and circles

Mathematics Admissions Test Questions

Question B from 2011 Paper

B. A rectangle has perimeter P and area A. The values P and A must satisfy

(a) $P^3 > A$, (b) $A^2 > 2P + 1$, (c) $P^2 \geqslant 16A$, (d) $PA \geqslant A + P$.

Solution:

B. Let the rectangle have sides x, y so that

$$2x + 2y = P, \qquad xy = A.$$

Eliminating y we see that

$$x^2 - \frac{1}{2}Px + A = 0.$$

As x is real it follows that the discriminant is non-negative and so

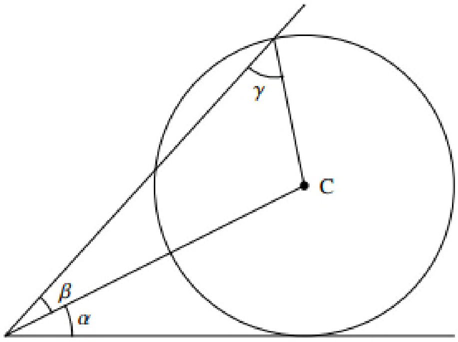
$$\left(\frac{P}{2}\right)^2 - 4 \times 1 \times A \geqslant 0 \implies P^2 \geqslant 16A.$$

The answer is (c). (This condition is in fact sufficient for x and y to be real and positive.)

NB. This is a good example of a question with an elegant solution that you should try to spot, but you are under time constraints and they do give you multiple choice for a reason; it is not cheating if you reach your answer by establishing that it can't be any of the other options. For example, arguably the simplest rectangle to consider is a 1 by 1 square. This has $P = 4$ and $A = 1$. For these values you immediately get that (b) and (d) aren't possible (check!). So it has to be (a) or (c). If we consider an x by x square then we get that (a) only holds if x is greater than 1/64 (check this as well!). So the answer has to be (c), and we have reached this conclusion without even venturing from squares to rectangles!

Question E from 2011 Paper

E. The circle in the diagram has centre C. Three angles α, β, γ are also indicated.



The angles α, β, γ are related by the equation:

(a) $\cos\alpha = \sin(\beta + \gamma)$;
(b) $\sin\beta = \sin\alpha \sin\gamma$;
(c) $\sin\beta\,(1 - \cos\alpha) = \sin\gamma$;
(d) $\sin(\alpha + \beta) = \cos\gamma \sin\alpha$.

Solution:

E. Without any loss of generality, we can assume the radius of the circle to be 1. Let A be the vertex of the angles α and β and B be the vertex of the angle γ. We see that AC has length $1/\sin\alpha$. Applying the sine rule to the triangle ABC we find

$$\frac{\sin\beta}{1} = \frac{\sin\gamma}{1/\sin\alpha}$$

and the answer is (b).

Question I from 2012 Paper

I. The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle's perimeter and area, respectively. Which of the following is true?

(a) $\frac{A}{P} = \frac{5}{4\pi}$; (b) $P < A$; (c) $\frac{P}{A} = \frac{10}{3\pi}$; (d) P^2 is rational.

Solution:

I. If the radius of the circle is r then we have $2\pi r = 10$ and $r = 5/\pi$. This distance r is also the distance from the centre of the triangle to any of its vertices. So we have

$$A = 3 \times \frac{1}{2} r^2 \sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{4} r^2; \qquad P = 6 \times r \sin\left(\frac{\pi}{3}\right) = 3\sqrt{3}r.$$

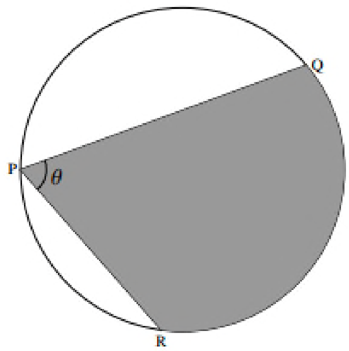
Hence

$$\frac{A}{P} = \frac{3\sqrt{3}r^2/4}{3\sqrt{3}r} = \frac{r}{4} = \frac{5}{4\pi}$$

and the answer is (a).

Question J from 2012 Paper

J. If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below,

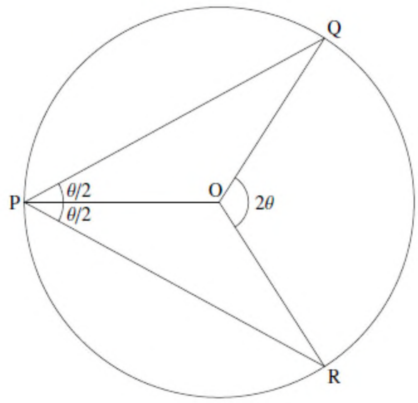


then the largest possible area of the shaded region RPQ is

(a) $\theta\left(1 + \cos\left(\frac{\theta}{2}\right)\right)$; (b) $\theta + \sin\theta$; (c) $\frac{\pi}{2}(1 - \cos\theta)$; (d) θ.

Solution:

J. The area QPR is largest when Q and R are symmetrically placed about P, for if (say) PQ were longer than PR then Q could be moved so as to gain more area than would be lost by the corresponding move of R. This means that the angles QPO and RPO are both $\theta/2$; the angle QOR is 2θ as the angle subtended by QR at the centre O is twice that subtended at the circumference P.



We then see that the area of PQR equals

$$\underbrace{2 \times \frac{1}{2} \times 1^2 \times \sin QOP}_{\text{triangles}} + \underbrace{\frac{1}{2} \times 1^2 \times 2\theta}_{\text{sector}} = \sin(\pi - \theta) + \theta = \sin\theta + \theta$$

and **the answer is (b).**