Year 12 Problem-Solving Course

Section 3: Number – prime factorisations, fractions and irrationals

Exercise - Sketch Solutions

1. $27 = 3^3$

   $147 = 3 \times 7^2$

   $27 \times 147 = 3^4 \times 7^2$

   $= (3^2 \times 7)^2$

   $= 63^2$

2. $\dfrac{10!}{2!4!4!} = \dfrac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times4\times3\times2\times1\times4\times3\times2\times1} = \dfrac{10\times9\times8\times7\times6\times5}{2\times4\times3\times2} = 10\times9\times7\times5$

   $\dfrac{10!}{3!3!4!} = \dfrac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1\times4\times3\times2\times1} = \dfrac{10\times9\times8\times7\times6\times5}{3\times2\times3\times2} = 10\times3\times4\times7\times5$

   $\dfrac{10!}{2!3!5!} = \dfrac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{2\times1\times3\times2\times1\times5\times4\times3\times2\times1} = \dfrac{10\times9\times8\times7\times6}{2\times3\times2} = 10\times9\times4\times7$

   It’s reasonably straightforward to see that

   $10 \times 3 \times 4 \times 7 \times 5 > 10 \times 9 \times 7 \times 5 > 10 \times 9 \times 4 \times 7$, since
   $$3 \times 4 \times 5 > 9 \times 5 > 9 \times 4$$

   (A slightly slicker way to do the above problem is to realise how much the three fractions ha
   $\frac{10!}{2!3!4!}$ then
   $\frac{10!}{2!4!4!} = \frac{X}{4}, \frac{10!}{3!3!4!} = \frac{X}{3}$ and $\frac{10!}{2!3!5!} = \frac{X}{5}$. You then immediately see that $\frac{X}{3} > \frac{X}{4} > \frac{X}{5}$.)

3. $91080 = 2 \times 45540 = 2^2 \times 22770 = 2^3 \times 11385$

   $= 2^3 \times 3 \times 3795 = 2^3 \times 3^2 \times 1265$

   $= 2^3 \times 3^2 \times 5 \times 253$

   $= 2^3 \times 3^2 \times 5 \times 11 \times 23$

   The number of factors is $4\times3\times2\times2\times2 = 96$, of which 94 are proper factors.

   The smallest proper factor of 91080 is 2 and the largest is 45540.

   $10^{20} = 2^{20} \times 5^{20}$

   The number of factors is $21\times21 = 441$, of which 439 are proper factors.

4. $3^2 \times 5^3$ has $4\times3 = 12$ factors, one is 1 and one is $3^2 \times 5^3$, leaving 10 proper factors.

The number of factors is $(m+1)\times(n+1)$, so $(m+1)\times(n+1)=12$

$mn+m+n+1=12$

$mn+m+n=11$

$m=1,\ n=5$

$m=2,\ n=3$

$m=3,\ n=2$

$m=4,\ n=\frac{7}{5}$

$m=5,\ n=1$

There are 4 integer solutions.

$3^3\times5^9$ has $4\times10=40$ factors and therefore 38 proper factors.
$5\times7^9\times11$ has $2\times10\times2=40$ factors and therefore 38 proper factors.

$40=2^3\times5$

This can be factorised into: $5\times2\times2\times2$

$5\times4\times2$

8×5

$10\times2\times2$

10×4

20×2

40

$5\times2\times2\times2$ gives $2^4\times3\times5\times7=1680$
$5\times4\times2$ gives $2^4\times3^3\times5=2160$
8×5 gives $2^7\times3^4=10368$
$10\times2\times2$ gives $2^9\times3\times5=7680$
10×4 gives $2^9\times3^3=13824$
20×2 gives $2^{19}\times3=1572864$
40 gives $2^{39}=549755813888$
The smallest number with 38 proper factors is 1680.

5. The number of zeros is determined by power of 10, which is equal to the number of 2×5 that can be formed.
The number of $\times5$ in $100!$ is 24: 1 for each multiple of 5 and an additional 1 for the 4 multiples of 25.
The number of $\times2$ is much larger as there are 50 multiples of 2 and additional $\times2$ for multiples of 4, 8, 16, 32, 64.
Therefore the number of $\times10$ is 24 and the number of zeros at the end is 24.

6. a) Let the four digit integer be x and the one digit integer y
$x=432y+2$

$y = 1 \; x = 434$

$y = 2 \; x = 866$

$y = 3 \; x = 1298$

b) The number is the smallest number that is a multiple of 2, 3, 5, 8, 13, 21, and 34 plus 1
The prime factors of those numbers are $2, 3, 5, 2^3, 13, 3\times 7, 2\times 17$
The prime factorisation of the smallest number that is a multiple of these is
$2^3 \times 3\times 5\times 7\times 13\times 17 = 185640$
The number is therefore 185641.

c) Adding all the equations together, $3a+3b+3c+3d = 5p$

$a+b+c+d = \dfrac{5p}{3}$

As $a+b+c+d$ is an integer, p must be a multiple of 3, and the only prime that is a multiple of 3 is 3, so $p = 3$

7. $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{a+c}{b+d}$.

$(b+d)\times(ad+bc) = abd+bcd$

$abd+bcd+b^2c+ad^2 = abd+bcd$

$b^2c+ad^2 = 0$

$b^2c = -ad^2$

The following integers satisfy this

$c = -1$, $b = 1$, $a = 1$, $d = 1$

This gives $\dfrac{1}{1}+\dfrac{-1}{1}=\dfrac{1+(-1)}{2}$.

There are (infinitely many) other possibilities such as:
$c = 3$, $d = 5$, $a = -12$, $b = 10$ giving $\dfrac{-12}{10}+\dfrac{3}{5}=\dfrac{-9}{15}$

8. For any real number x and integer n, $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ – this is true, adding an integer to a number will result in its floor changing by that integer.

For any real numbers x and y, $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ – this is false. For example if $x = 3.7$ and $y = 3.4$ then $\lfloor x \rfloor = 3$ and $\lfloor y \rfloor = 3$ but $\lfloor x+y \rfloor = 7$.

For any real numbers λ and x, $\lfloor \lambda x \rfloor = \lambda \lfloor x \rfloor$ – this is false. For example if $x = 1.4$ and $\lambda = 3$ then $\lfloor x \rfloor = 1, \lambda \lfloor x \rfloor = 3$ but $\lambda x = 4.2$ and $\lfloor \lambda x \rfloor = 4$.

For any real numbers x and y, $\lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$ – this is false, take $x = 1.4$ and $y = 3$.

For any real number x, $\lfloor\lfloor x\rfloor\rfloor = \lfloor x\rfloor$ – this is true because taking the floor of a number that is already an integer (like $\lfloor x\rfloor$) will not change it.

9. a) Suppose x satisfies

$$\lfloor x\rfloor = x - \lfloor x\rfloor$$

Then $2\lfloor x\rfloor = x$ and since $\lfloor x\rfloor$ is an integer this means that x is an integer. Therefore $\lfloor x\rfloor = x$ and $x - \lfloor x\rfloor = 0$. Since $\lfloor x\rfloor = x - \lfloor x\rfloor$, it follows that $[x] = 0$. Remembering that x is an integer, the only possibility is that $x = 0$.

b) Suppose x satisfies

$$\lfloor\sqrt{x}\rfloor = \sqrt{\lfloor x\rfloor}$$

Then $\lfloor\sqrt{x}\rfloor^2 = \lfloor x\rfloor$. This means that $\lfloor x\rfloor$ is the square of a whole number and so $n^2 \le x < n^2 + 1$ for some integer n (we can suppose that n is positive).

If $n^2 \le x < n^2 + 1$ then $n \le \sqrt{x} < \sqrt{(n^2 + 1)}$. This means that $\lfloor\sqrt{x}\rfloor = n$ and $\sqrt{\lfloor x\rfloor} = \sqrt{n^2} = n$.

Therefore the solutions of the equations are the real numbers x that satisfy $n^2 \le x < n^2 + 1$ for some integer n.

10. The statement is false, $a = \sqrt[4]{2},\ b = \sqrt{2}$ is counter example. Note that $\sqrt[4]{2}$ is irrational, if it were not then its square which is $\sqrt{2}$ would be rational.

11. If p and q are both odd then $p^3 + pq^2 + q^3$ is odd (which it isn't).
‘p odd and q even’ and ‘p even and q odd’ lead to similar contradictions.

Therefore p and q are both even and p/q is not in ‘lowest terms’.

Second part follows immediately since if $\left(\frac{p}{q}\right)^3 + \frac{p}{q} + 1 = 0$ then $p^3 + pq^2 + q^3 = 0$.

12. $a)$

$$\frac{1}{3+\sqrt{8}} = \frac{3-\sqrt{8}}{(3+\sqrt{8})\times(3-\sqrt{8})} = \frac{3-\sqrt{8}}{1} = 3-\sqrt{8}$$

$b)$

$$\frac{17}{100} < \frac{1}{3+\sqrt{8}} < \frac{43}{250}$$

$$\Leftrightarrow \frac{17}{100} < 3-\sqrt{8} < \frac{43}{250}$$

$$\Leftrightarrow \frac{-283}{100} < -\sqrt{8} < \frac{-707}{250}$$

$$\Leftrightarrow \frac{283}{100} > \sqrt{8} > \frac{707}{250}$$

$$\Leftrightarrow \frac{80089}{10000} > 8 > \frac{499849}{62500}$$

$$\Leftrightarrow 8.0089 > 8 > 7.997584$$

The final statement is clearly true. Since every logical step of the working worked in both directions, it follows that our original statement is also true.

c) $\frac{283}{100} > \sqrt{8} > \frac{707}{250}$ (from our working for (b))

$$\frac{283}{100} = 2.83 \quad \frac{707}{250} = 2.828$$

$$2.83 > \sqrt{8} > 2.828$$

$$\sqrt{8} = 2.83 \text{ (2dp)}$$