Year 12 Problem-Solving Course

Section 3: Number - prime factorisations and irrationals

Mathematics Admissions Test Questions

Question A from the 2007 Paper

A. Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

is an integer if

(a) $r + s \leqslant 0$,
(b) $s \leqslant 0$,
(c) $r \leqslant 0$,
(d) $r \geqslant s$.

Solution:

A: Separating out the powers of 2 and 3 we have

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}} = 2^{(r+s+2r-2s-3r)} \times 3^{(r+s+r-s-2r-4s)} = 2^{-s} \times 3^{-4s}$$

which is an integer if $s \leqslant 0$. **The answer is (b).**

Question B from the 2012 Paper

B. Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers. Then N will definitely be a square number whenever

(a) k is even;
(b) $k + n$ is odd;
(c) k is odd but $m + n$ is even;
(d) $k + n$ is even.

Solution:

B. We can rewrite N as

$$N = 2^k \times 4^m \times 8^n = 2^{k+2m+3n}.$$

Now 2^r is a square when r is even and is not a square when r is odd. So we need

$$k + 2m + 3n = k + n + 2(m + n)$$

to be even , which is equivalent to needing $k + n$ to be even. **The answer is (d).**

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Question 2 from the 2010 Paper

2. For **ALL APPLICANTS.**

Suppose that a, b, c are *integers* such that

$$a\sqrt{2} + b = c\sqrt{3}.$$

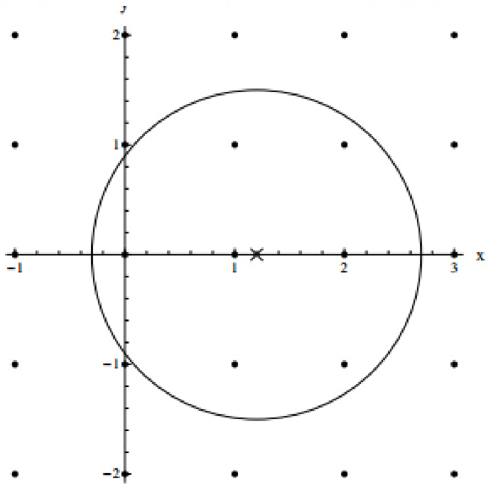
(i) By squaring both sides of the equation, show that $a = b = c = 0$.

[You may assume that $\sqrt{2}, \sqrt{3}$ and $\sqrt{2/3}$ are all irrational numbers. An irrational number is one which cannot be written in the form p/q where p and q are integers.]

(ii) Suppose now that m, n, M, N are *integers* such that the distance from the point (m, n) to $(\sqrt{2}, \sqrt{3})$ equals the distance from (M, N) to $(\sqrt{2}, \sqrt{3})$.

Show that $m = M$ and $n = N$.

Given real numbers a, b and a positive number r, let $N(a, b, r)$ be the number of integer pairs x, y such that the distance between the points (x, y) and (a, b) is less than or equal to r. For example, we see that $N(1.2, 0, 1.5) = 7$ in the diagram below.



(iii) Explain why $N(0.5, 0.5, r)$ is a multiple of 4 for any value of r.

(iv) Let k be any positive integer. Explain why there is a positive number r such that

$$N\left(\sqrt{2}, \sqrt{3}, r\right) = k.$$

Solution:

2. (i) [4 marks] If $a\sqrt{2} + b = c\sqrt{3}$ then squaring both sides of the equation gives

$$2a^2 + b^2 + 2ab\sqrt{2} = 3c^2.$$

If $ab \neq 0$ then

$$\sqrt{2} = \frac{3c^2 - 2a^2 - b^2}{2ab}$$

is rational – a contradiction and so $a = 0$ or $b = 0$. If $a = 0$ then we have $\sqrt{3} = b/c$ unless $b = c = 0$; if $b = 0$ then we have $\sqrt{2/3} = c/a$ unless $c = a = 0$.

(ii) [4 marks] We have that the square of the distance from (m, n) to $(\sqrt{2}, \sqrt{3})$ equals the square of the distance from (M, N) to $(\sqrt{2}, \sqrt{3})$, or put algebraically

$$\left(m - \sqrt{2}\right)^2 + \left(n - \sqrt{3}\right)^2 = \left(M - \sqrt{2}\right)^2 + \left(N - \sqrt{3}\right)^2. \tag{1}$$

Rearranging this gives

$$2\left(M - m\right)\sqrt{2} + \left(m^2 + n^2 - M^2 - N^2\right) = 2\left(n - N\right)\sqrt{3}.$$

By part (i) we have that $2\left(M - m\right) = 0 = 2\left(n - N\right)$ and hence $M = m$ and $N = n$.

(iii) [4 marks] If a particular point (x, y) is within distance r of $\left(\frac{1}{2}, \frac{1}{2}\right)$ then so will its reflection in the $x = \frac{1}{2}$ line, the $y = \frac{1}{2}$ line and in both lines as these are diameters of the circle. The coordinates of the three points are integers also. Precisely these are the points

$$\left(1 - x, y\right), \qquad \left(x, 1 - y\right), \qquad \left(1 - x, 1 - y\right).$$

As the lattice points within the circle can be divided into sets of four like above then $N\left(\frac{1}{2}, \frac{1}{2}, r\right)$ is a multiple of 4.

(iv) [3 marks] As r increases then the circle (and its boundary) consumes lattice points. But because no two lattice points are equidistant from $\left(\sqrt{2}, \sqrt{3}\right)$ – as shown in part (ii) – then the lattice points are consumed one at a time and all positive integers are achieved by $N\left(\sqrt{2}, \sqrt{3}, r\right)$.