Degree Topics in Mathematics

Cramer's Rule

To study this topic, you will need to be familiar with matrices and how to find the determinant of a 2x2 or a 3x3 matrix.

If you are not familiar with matrices and determinants, or need to revise them the following links might be helpful:

- **A basic reminder of matrices from Maths Is Fun**
- **Finding the determinant of a 2x2 or 3x3 matrix**
- **An introduction to matrices and determinants from the University of Surrey**

Cramer's Rule is a method used to find the solution of a set of equations. It is important that there are the same number of variables as there are equations, e.g. two equations in two unknown variables x and y.

We can express a system of equations in matrix form. For example:

$$2x + 5y = 11$$

$$3x - 4y = 28$$

can be expressed in the form $\left[\begin{array}{cc|c} 2 & 5 & 11 \\ 3 & -4 & 28 \end{array}\right]$

Matrix of coefficients

Column vector of constants

We will call the matrix on the left-hand side of the dotted line A, and the column vector on then right-hand side of the dotted line b.

Cramer's Rule

A simple statement of Cramer's rule is:

- Find the determinant of matrix A
- Replace column 1 of matrix A with column b. Call this matrix A_1
- Find the determinant of matrix A_1
- Repeat these two steps for each column of A

Then the solution to the equations is:

$$x_1 = \frac{\det(A_1)}{\det(A)}\,, \qquad x_2 = \frac{\det(A_2)}{\det(A)}\,, \quad etc\ \ldots$$

These stages are often expressed more formally in undergraduate textbooks, for example in this straightforward **guide** for Engineering University from Sheffield University.

Example

Use Cramer's Rule to solve the equations:

$$2x + 5y = 11$$
$$3x - 4y = 28$$

Formulating the equations in an augmented matrix we get:

$$\left[\begin{array}{cc|c} 2 & 5 & 11 \\ 3 & -4 & 28 \end{array}\right]$$

So A = $\begin{bmatrix} 2 & 5 \\ 3 & -4 \end{bmatrix}$ and b = $\begin{bmatrix} 11 \\ 28 \end{bmatrix}$

det (A) can be written $\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix}$. This is calculated as (2 x -4) – (3 x 5) = -23

Replacing the first column of A by column b gives A_1= $\begin{bmatrix} 11 & 5 \\ 28 & -4 \end{bmatrix}$ which has determinant

(11 x -4) – (5 x 28) = -184

Replacing the second column of A by column b gives A_2 = $\begin{bmatrix} 2 & 11 \\ 3 & 28 \end{bmatrix}$ which has determinant

(2 x 28) – (3 x 11) = 23

Then $x = \frac{-184}{-23} = 8$ and $y = \frac{23}{-23} = -1$.

The answer can be checked by substituting the values of x and y back into the original equations.

Task 1

Use Cramer's Rule to solve the equations:

$$7x - 2y = 3$$
$$3x + y = 5$$

Task 2

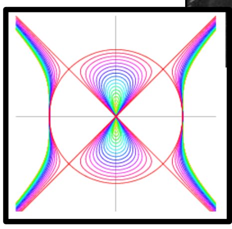
Use Cramer's Rule to solve the equations:

$$3x - 5y = 26$$
$$5x + y = 6$$

Who was Cramer?

Gabriel Cramer (1704-1752) was a Swiss mathematician and although he isn't necessarily remembered as one of the greatest mathematicians of his day, he is well regarded as a disseminator of mathematical ideas. Cramer is remembered for his statement of Cramer's Rule (for solving equations), which is outlined in this worksheet, and he also extensively studied Devil's curves, which are curves of the form:

$y^2(y^2 - a^2) = x^2(x^2 - b^2)$ (see image, right)



Images taken from http://www-history.mcs.st-and.ac.uk/PictDisplay/Cramer.html and http://mathworld.wolfram.com/DevilsCurve.html

Cramer's rule can be used for n equations in n unknowns. This means we can use it to solve 3 equations in 3 unknowns, as long as we can find the determinant of a 3x3 matrix.

The **determinant of a 3x3 matrix** $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ can be found using the formula:

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Task 3

Use Cramer's Rule to solve the equations:

$$x + y + z = 5$$

$$x - 2y - 3z = -1$$

$$2x + y - z = 3$$

Task 4

Use Cramer's Rule to solve the equations:

$$4x + 5y \quad = 2$$

$$11x + y + 2z = 3$$

$$x + 5y + 2z = 1$$

Fancy a challenge?

Look at the **Maths is Fun website** for instructions about how to find the determinant of a 4x4 matrix.

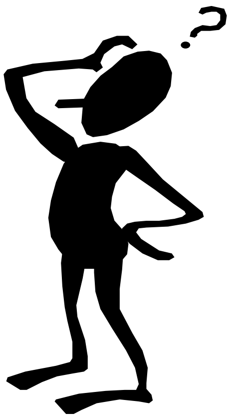
Use this method and Cramer's Rule to solve the equations:

$$4x + y + z + w = 6$$

$$3x + 7y - z + w = 1$$

$$7x + 3y - 5z + 8w = -3$$

$$x + y + z + 2w = 3$$



Solutions

Task 1

det (A) = $\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13$, det ($A_1$) = $\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13$, det($A_2$) = $\begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$

So $x = \frac{13}{13}$ = 1, $y = \frac{26}{13} = 2$

Task 2

det (A) = $\begin{vmatrix} 3 & -5 \\ 5 & 1 \end{vmatrix} = 28$, det ($A_1$) = $\begin{vmatrix} 26 & -5 \\ 6 & 1 \end{vmatrix} = 56$, det($A_2$) = $\begin{vmatrix} 3 & 26 \\ 5 & 6 \end{vmatrix} = -112$

So $x = \frac{56}{28}$ = 2, $y = \frac{-112}{28} = -4$

Task 3

det(A) = $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = 5$, det($A_1$) = $\begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix} = 20$, det($A_2$) = $\begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix} = -10$ and det(A_3) = $\begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 15$

Therefore $x = \frac{20}{5} = 4, y = \frac{-10}{5} = -2, z = \frac{15}{5} = 3$

Task 4

det(A) = $\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -132$, det($A_1$) = $\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -36$, det($A_2$) = $\begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -24$ and det(A_3) = $\begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 12$

Therefore $x = \frac{-36}{-132} = \frac{3}{11}, y = \frac{-24}{-132} = \frac{2}{11}, z = \frac{12}{-132} = -\frac{1}{11}$

Fancy a challenge?

det(A) = $4\begin{vmatrix} 7 & -1 & 1 \\ 3 & -5 & 8 \\ 1 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 & 1 \\ 7 & -5 & 8 \\ 1 & 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 7 & 1 \\ 7 & 3 & 8 \\ 1 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 7 & -1 \\ 7 & 3 & -5 \\ 1 & 1 & 1 \end{vmatrix}$

$= 4(-120) - 1(-36) + 1(-44) - 1(-64)$

= - 424

det(A_1) = $6\begin{vmatrix}7 & -1 & 1\\3 & -5 & 8\\1 & 1 & 2\end{vmatrix} - 1\begin{vmatrix}1 & -1 & 1\\-3 & -5 & 8\\3 & 1 & 2\end{vmatrix} + 1\begin{vmatrix}1 & 7 & 1\\-3 & 3 & 8\\3 & 1 & 2\end{vmatrix} - 1\begin{vmatrix}1 & 7 & -1\\-3 & 3 & -5\\3 & 1 & 1\end{vmatrix}$

$= 6(-120) - 1(-36) + 1(196) - 1(-64)$

= - 424

det(A_2) = $4\begin{vmatrix}1 & -1 & 1\\-3 & -5 & 8\\3 & 1 & 2\end{vmatrix} - 6\begin{vmatrix}3 & -1 & 1\\7 & -5 & 8\\1 & 1 & 2\end{vmatrix} + 1\begin{vmatrix}3 & 1 & 1\\7 & -3 & 8\\1 & 3 & 2\end{vmatrix} - 1\begin{vmatrix}3 & 1 & -1\\7 & -3 & -5\\1 & 3 & 1\end{vmatrix}$

$= 4(-36) - 6(-36) + 1(-72) - 1(0)$

= 0

det(A_3) = $4\begin{vmatrix}7 & 1 & 1\\3 & -3 & 8\\1 & 3 & 2\end{vmatrix} - 1\begin{vmatrix}3 & 1 & 1\\7 & -3 & 8\\1 & 3 & 2\end{vmatrix} + 6\begin{vmatrix}3 & 7 & 1\\7 & 3 & 8\\1 & 1 & 2\end{vmatrix} - 1\begin{vmatrix}3 & 7 & 1\\7 & 3 & -3\\1 & 1 & 3\end{vmatrix}$

$=4(-196) - 1(-72) + 6(-44) - 1(-128)$

= - 848

det(A_4) = $4\begin{vmatrix}7 & -1 & 1\\3 & -5 & -3\\1 & 1 & 3\end{vmatrix} - 1\begin{vmatrix}3 & -1 & 1\\7 & -5 & -3\\1 & 1 & 3\end{vmatrix} + 1\begin{vmatrix}3 & 7 & 1\\7 & 3 & -3\\1 & 1 & 3\end{vmatrix} - 6\begin{vmatrix}3 & 7 & -1\\7 & 3 & -5\\1 & 1 & 1\end{vmatrix}$

$=4(-64) - 1(0) + 1(-128) - 6(-64)$

= 0

So $x = \frac{-424}{-424} = 1, y = \frac{0}{-424} = 0, z = \frac{-848}{-424} = 2, w = \frac{0}{-424} = 0$

You can check these values are correct by substituting the values back into the original equations.