The Further Mathematics Support Programme

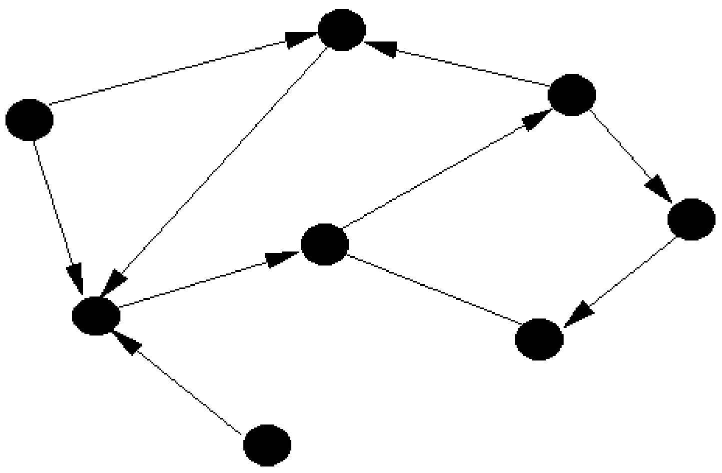
Graph Theory

With kind permission, this resource contains materials from the online **OR-notes** by J. E. Beasley.

Graph theory deals with problems that have a graph (or network) structure. In this context a graph consists of:

- **vertices** or **nodes** - which are a collection of points; and
- **arcs** - which are lines running between the nodes. Such arcs may be **directed** (indicated by an arrow) or undirected.

If you have studied Decision / Discrete Mathematics at A level, you will be familiar with graphs. An example of a directed graph is shown below:

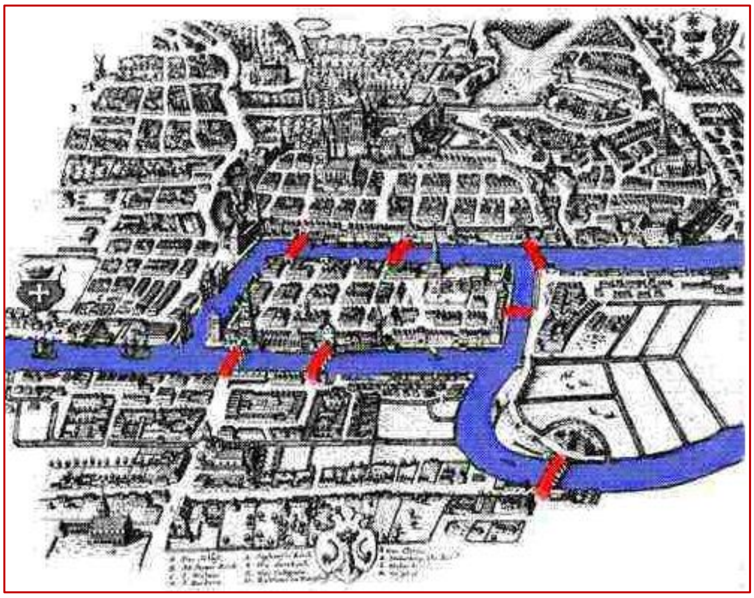


Graph theory is used in dealing with problems which have a natural graph structure, for example:

- road networks where nodes = towns/road junctions, arcs = roads
- communication networks such as telephone systems
- computer systems

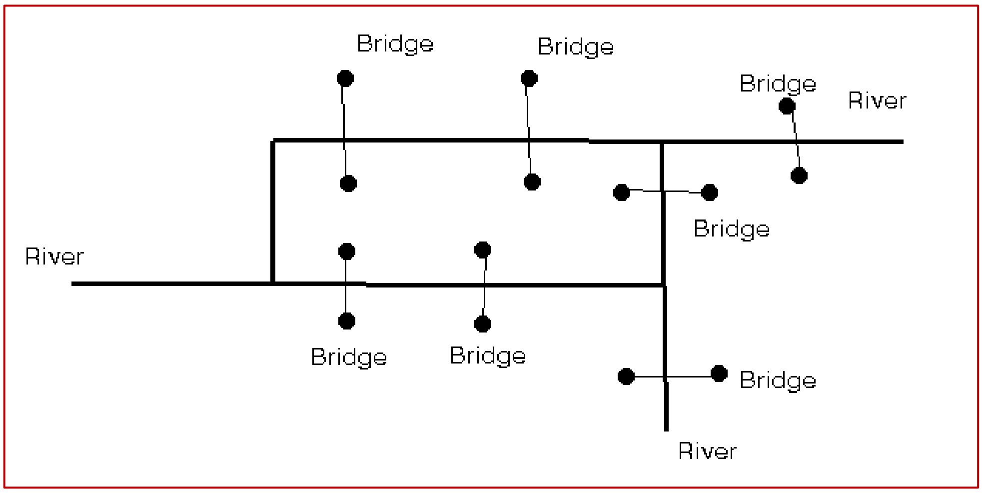
A famous graph problem

In 1736 Euler solved the problem of whether, given the map below of the city of Konigsberg (now called Kalingrad) in Russia, someone could make a complete tour, crossing over all 7 bridges over the river Pregel, and return to their starting point without crossing any bridge more than once.



(map taken from **The MacTutor History of Mathematics Archive**)

What was Euler's conclusion? The picture below shows the city, but simplified so that just the river and bridges are shown. Do you think that someone could make a complete tour, crossing over all 7 bridges over the river, and return to their starting point without crossing any bridge more than once?



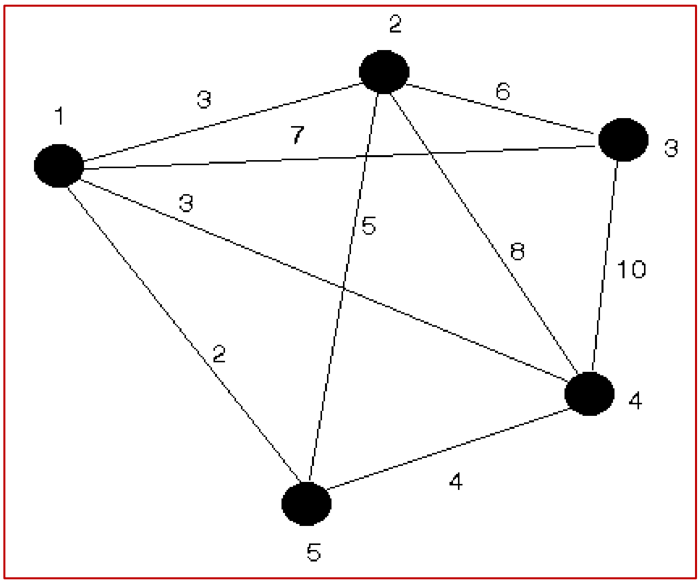
More about the Konigsberg Bridge Problem and the traversibility of graphs can be found **here**.

Task 1

In the diagram shown below there are four wells in an offshore oilfield (nodes 1 to 4 below) and an on-shore terminal (node 5 below). The four wells in this field must be connected together via a pipeline network to the on-shore terminal. Note that not all wells need to be directly linked to the terminal – they can be connected via another well.

The various pipelines that can be constructed are shown as arcs in the diagram below and the cost of each pipeline is given on each arc.

What pipelines would you recommend be built?



Try this task before reading any further.

Minimum Connector Problems

Task 1 is a specific example of a more general problem - namely given a graph such as that shown above, which arcs would we use so that:

- the total cost of the links used is a minimum; and
- all the points are connected.

The solution is called the *minimum spanning tree (MST).*

For example, in the diagram above, one possible structure connecting all the points together would consist of the arcs 1-2, 2-3, 3-4, 4-5 and 5-1 with a total cost of 25, but there are other structures where the total cost of the arcs used is smaller. For example, we currently have a cycle (i.e. a closed path around the nodes 1 to 5). Removing arc 3-4 is a good choice as it is an expensive arc and we can reduce the cost to 15, whilst still maintaining a link from the on-shore terminal to the offshore oil fields. Removing cycles ensures we are left with a **tree** (a connected graph with no cycles). Note that for a graph with n nodes, the MST will have $n-1$ arcs.

Kruskal's Algorithm for finding a MST

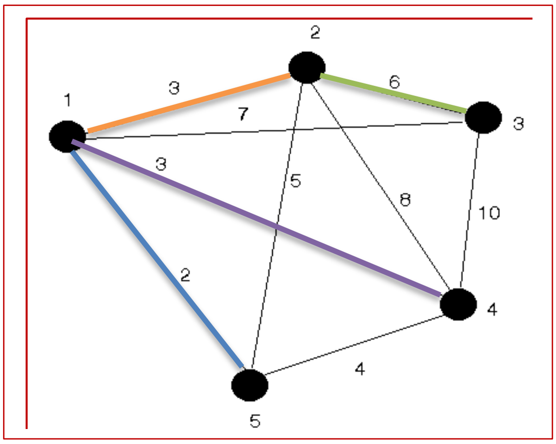
The algorithm is:

> "For a graph with n nodes keep adding the shortest (least cost) arc - avoiding the creation of cycles - until $n-1$ links have been added"

Note here that the Kruskal algorithm only applies to graphs in which all the links are *undirected*.

For the graph in Task 1, we can list the arcs in ascending order and indicate whether or not they will be added to the tree. It is helpful to show this in a table and highlight the included arcs on the diagram.

Arc	Cost	Decision
1-5	2	add to tree (blue)
1-2	3	add to tree (orange)
1-4	3	add to tree (purple)
5-4	4	reject as forms cycle 1-5-4-1
5-2	5	reject as forms cycle 1-5-2-1
2-3	6	add to tree (green)



Stop as four arcs have been added and these are all we need for the MST of a graph with 5 nodes. Note that arcs 1-2 and 1-4 could have been listed the other way around as they are the same cost.

Hence the MST for Task 1 consists of the arcs 1-5, 1-2, 1-4 and 2-3 with total cost 14.

A pioneering application of MST models to pipeline network design by Frank and Frisch occurred in the early 1960's, in the design of a gas pipeline network in the Gulf of Mexico. Savings resulted of the order of tens of millions of dollars. This illustrates how, in OR, theoretical problems like the MST can be used as building blocks to solve complex practical problems.

Task 2

Further practice with Kruskal's algorithm and an introduction to Prim's algorithm (which also finds a MST) can be found in this **Nuffield Foundation** activity (with answers available in the associated **Teacher Guide**).

Task 3

Research some of the following common algorithms that involve graphs:

- **Eulerian graphs and Hamiltonian cycles** – Eulerian graphs are used in the Konigsberg bridges problem and Hamiltonian cycles are used in the solution of Travelling Salesman type problems.
- **Flows** – there are many situations in which substances flow through a network of pipes, for example water through water pipe systems. Identifying the maximum amount of the substance that can flow through a particular system is vital. The maximum-flow-minimum cut algorithm is used here.
- **Dijkstra's Algorithm** – used to find the shortest route between two vertices in a network
- **Planarity algorithm** – a graph is planar if it can be drawn on a two-dimensional surface without any of the arcs crossing each other. These are important in electrical engineering when designing electrical circuits, for example. A common algorithm uses Hamiltonian cycles.
- **Matchings** – allocating individuals to tasks can be carried out using a specific type of graph called a **bipartite graph**. These graphs consist of two separate sets of nodes, with arcs passing from nodes in one set to nodes in the other.