FMSP | Let maths take you further

The Further Mathematics Support Programme

Hyperbolic Functions

If you have studied Further Mathematics A level, you will already have met the hyperbolic functions. If not, this worksheet will introduce you to the main ideas.

The hyperbolic functions have some similarities to the trigonometric functions $\sin x$, $\cos x$ and $\tan x$ and are sometimes called the 'hyperbolic sine', 'hyperbolic cosine' and 'hyperbolic tan'.

The three main hyperbolic functions are:

- $sinh\, x$ (pronounced 'shine x')
- $cosh\, x$ (pronounced 'cosh x')
- $tanh\, x$ (pronounced 'tansh x')

The hyperbolic functions are defined in terms of exponential functions, with $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

From A level Mathematics you will recall the following identities:

$$tanx = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x} \qquad cosec\ x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

Task 1

The identities in the box above are also true for hyperbolic functions.

Use the identities and the definitions of $sinh\, x$ and $cosh\, x$ to find identities for:

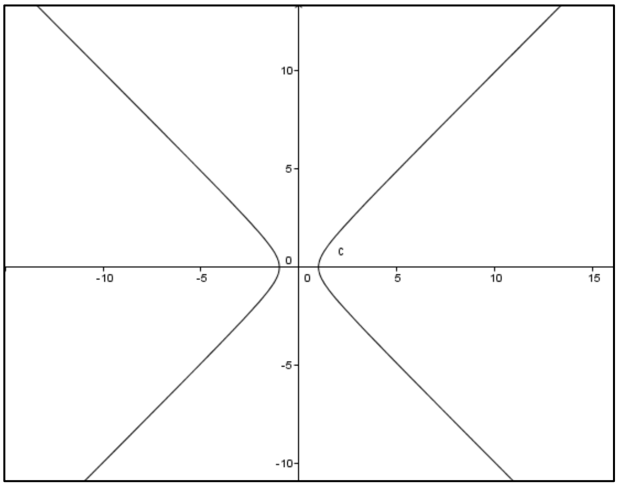
a) $tanh\, x$

b) $sech\, x$ (pronounced 'sesh x')

c) $cosech\, x$ (pronounced 'cosesh x')

d) $coth\ \ x$ (pronounced 'coth x')

At this point it is worth taking a moment to consider why these functions are called 'hyperbolic' functions. Hyperbolae are a family of curves which have similar shapes to each other, in a similar way to the family of parabolas. A basic hyperbola is the curve $y = \frac{1}{x}$ which you will recall has two asymptotes (the x and y axes). Another hyperbola is $x^2 - y^2 = 1$ which is pictured below.



The curve has two asymptotes: $y = x$ and $y = -x$ (not pictured).

Task 2

Consider the parametric equations $x = cosh\, t\,, y = sinh\, t$

Using the definitions of $cosh\, t$ and $sinh\, t$ show that the parametric equations $x = cosh\, t\,, y = sinh\, t$ satisfy the equation $x^2 - y^2 = 1$

There are lots of identities for hyperbolic functions and many of these are similar to the trigonometric identities, often with a sign change.

Task 3

Using the definitions of the functions $cosh\, x$ and $sinh\, x$ prove the identity:

$$cosh^2 x - sinh^2 x \equiv 1$$

From this identity, deduce the identities:

$$1 - tanh^2 x \equiv sech^2 x$$

$$coth^2 x - 1 \equiv cosech^2 x$$

Task 4

(a) Prove the identity:

$$\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$$

and use it to prove the identity:

$$\cosh 2x \equiv cosh^2 x + sinh^2 x$$

(b) Use the definition of $\sinh x$ to prove the identity:

$$\sinh 2x \equiv 2 \sinh x \cosh x$$

(c) Using your answers to parts (a) and (b), find an identity for $\tanh 2x$ in terms of $\tanh x$

Osborn's Rule

As you worked with the identities above, you may have noticed some patterns. Here are some trigonometric and hyperbolic identities:

Trigonometric identities	Hyperbolic identities
$cos^2 x + sin^2 x \equiv 1$	$cosh^2 x - sinh^2 x \equiv 1$
$\cos 2x \equiv 1 - 2sin^2 x$	$\cosh 2x \equiv 1 + 2sinh^2 x$
$\sin 2x \equiv 2sinx \cos x$	$\sinh 2x \equiv 2 \sinh x \cosh x$

In the first two rows, a sign change occurs. In both cases this is where a $sin^2 x$ term becomes a $sinh^2 x$ term. A sign change does not occur in the third row, as the $\sin x$ term is not squared.

Osborn's Rule states that when changing from a trigonometric identity to a hyperbolic identity, a sign change occurs every time there is a product (or an implied product) of sines.

So, for example $1 + tan^2 x \equiv sec^2 x$ becomes $1 - tanh^2 x \equiv sech^2 x$ because $tan^2 x$ is an implied product of primes as it can be written as $\frac{sin^2 x}{cos^2 x}$

To learn more about hyperbolic functions and to practice further examples, including solving equations involving hyperbolic functions, see this useful **Centre for Innovation in Mathematics Teaching** textbook chapter.

Solutions

Task 1

a) $tanh\,x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$ and multiplying through the numerator and denominator by e^x gives $\frac{e^{2x}-1}{e^{2x}+1}$

Note: $e^x \times e^{=x} = 1$ and $e^x \times e^x = e^{2x}$

b) $sech\,x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

c) $cosech\,x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

d) $coth\,x = \frac{1}{\tanh x} = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$ or $\frac{e^{2x}+1}{e^{2x}-1}$

Task 2

$x = cosh\,t = \frac{1}{2}(e^t + e^{-t}) \quad y = sinh\,t = \frac{1}{2}(e^t - e^{-t})$

So $x^2 - y^2 = \left[\frac{1}{2}(e^t + e^{-t})\right]^2 - \left[\frac{1}{2}(e^t - e^{-t})\right]^2$

$= \frac{1}{4}(e^{2t} + 2 + e^{-2t}) - \frac{1}{4}(e^{2t} - 2 + e^{-2t})$

$= \frac{1}{4}(4)$

=1

Hence $x = cosh\,t$ and $y = sinh\,t$ satisfy the equation of the hyperbola.

This is why they are known as hyperbolic functions.

Task 3

$cosh^2x - sinh^2x \equiv \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$

$$\equiv \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$\equiv \tfrac{1}{4}(4) \equiv 1$$

$$1 - tanh^2x \equiv \ 1 - \frac{sinh^2x}{cosh^2x} \equiv \frac{cosh^2x - sinh^2x}{cosh^2x} \equiv \frac{1}{cosh^2x} \equiv sech^2x$$

$$coth^2x - 1 \equiv \frac{cosh^2x}{sinh^2x} - 1 \equiv \frac{cosh^2x - sinh^2x}{sinh^2x} \equiv \frac{1}{sinh^2x} \equiv cosech^2x$$

Task 4

(a)

We need to prove: $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$

Right hand side $\equiv \left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right) + \left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)$

$$\equiv \tfrac{1}{4}(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})$$

$$\equiv \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y})$$

$$\equiv \tfrac{1}{2}(e^{x+y} + e^{-(x+y)})$$

$$\equiv \cosh(x+y)$$

If we let $x = y$ then $\cosh(x+x) \equiv \cosh x \cosh x + \sinh x \sinh x$

which simplifies to $\cosh 2x \equiv cosh^2x + sinh^2x$

(b)

$$\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x})$$

$e^{2x} - e^{-2x}$ is a difference if two squares and can be expressed $(e^x - e^{-x})(e^x + e^{-x})$ and therefore

$$\sinh 2x \equiv \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$$

$$\equiv 2 \times \left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right) \equiv 2\sinh x \cosh x$$

(c)

$$\tanh 2x \equiv \frac{\sinh 2x}{\cosh 2x} \equiv \frac{2\sinh x \cosh x}{cosh^2 x + sinh^2 x}$$

Dividing numerator and denominator by $cosh^2 x$ gives:

$$\tanh 2x \equiv \frac{\left(\frac{2\sinh x}{\cosh x}\right)}{1 + tanh^2 x} \equiv \frac{2\tanh x}{1 + tanh^2 x}$$