The Further Mathematics Support Programme

Kinematics

Kinematics is the study of the motion of an object. It is a common topic in Mechanics units at A level, but if you have not studied any mechanics this worksheet will provide a useful introduction to the topic.

To simplify the study of kinematics, it is common to start by considering situations which have certain **modelling assumptions**:

- The object is a **particle** – this means that all the mass is concentrated at one point and so we do not have to consider any spinning of the object or any air resistance;

- The object moves with **constant acceleration**. If moving vertically, the object moves with constant **acceleration due to gravity**. This rate of acceleration is denoted g and has a value of approximately 9.8m/s^2.

Later in the study of kinematics, these assumptions are considered in greater detail to model the movement of the object more accurately.

What is meant by 'acceleration due to gravity'?

The story goes that Issac Newton was sitting under a tree one day when an apple fell from the tree and landed on him. He started to think about why the apple fell, why it was attracted to the ground and how far 'upwards' gravity extended. His Theory of Universal Gravitation was published in 1687.

Gravity gives weight to physical objects – that is why a person walking on the moon seems to 'float' whereas on Earth they stay on the ground. The gravitational acceleration on Earth is 9.80665m/s^2 whereas on the moon it is only 1.622m/s^2, meaning things fall towards the surface much more slowly.

In fact, there are slight differences in the gravitational acceleration at different points on the Earth's surface, from 9.78m/s^2 near the equator to 9.83m/s^2 near the poles, but we tend to work with 9.8m/s^2 in mathematics.

Image taken from http://www.grouporigin.com/clients/qatarfc

It is also important to note that acceleration due to gravity acts towards the surface of the earth and so even objects thrown upwards will accelerate towards the ground, meaning that as soon as they are thrown their upwards speed will be reducing i.e. they will *decelerate* at a rate of 9.8m/s^2.

You can read more about the 'real' story of Newton and his apple in **New Scientist online**.

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What situations can be studied using kinematics?

The speed camera problem: A car accelerates from rest at a set of traffic lights and passes through a speed camera at a distance of 50m from the traffic lights.

If the car's acceleration is 1.5 m/s^2, is the car exceeding the speed limit of 40km/h when it passes the camera?

The falling book problem: A book falls from a shelf to the floor, taking 0.45 seconds to land.

How high is the shelf?

Playing catch: If a ball is thrown vertically upwards with a speed of 15m/s and is caught in the same position 5 seconds later, find the maximum height of the ball above its starting point.

We'll return to these problems later. Firstly, the formulae used in kinematics will be considered:

The following symbols are used in kinematics:

s	The **displacement** of the object from its starting point. This is similar to distance but also takes account of the direction moved – on a straight line, for example, one direction is chosen as positive and the opposite direction as negative.
u	The **initial velocity** of the object. Velocity is similar to speed, but again the direction is taken into account – on a straight line one direction is chosen as positive and the opposite direction as negative.
v	The **final velocity** of the object. This might be at the end of the movement of the object or at one particular instant when we want to 'pause' the motion and see what is happening at that time.

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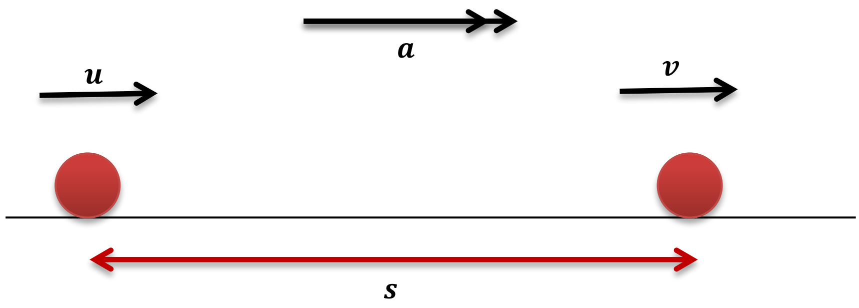
a	The **acceleration** of the object. If the object moves freely under gravity, a=9.8m/s^2
t	The **time** taken to complete the motion.

The equations that are used to solve kinematics problems are often called $suvat$ equations (see first column of the table). When using these equations we must have a situation where there is **constant** or **uniform acceleration**.



Note the use of a single headed arrow to represent velocity and a double headed arrow to represent acceleration.

This is standard notation.

To find the first kinematics ($suvat$) equation, use the formula:

$$acceleration = \frac{change\ in\ velocity}{time}$$

In this case, the equation becomes $a = \frac{v-u}{t}$ which simplifies to: $\boldsymbol{v = u + at}$

The situation shown in the diagram above can also be considered using the following formula:

$$distance\ moved = average\ speed \times time$$

In this case, $s = \left(\frac{u+v}{2}\right)t$ or $\boldsymbol{s = \frac{1}{2}(u+v)t}$

Task 1

The other three kinematics formulae are:

$$v^2 = u^2 + 2as$$

$$s = ut + \frac{1}{2}at^2$$

$$s = vt - \frac{1}{2}at^2$$

Each of these equations can be obtained by eliminating different variables in the first two kinematics equations. Carry out the rearrangements, stating which variable you are eliminating in each case.

To use the $suvat$ equations, it is helpful to list the information that has been provided in then question and identify which variable is required.

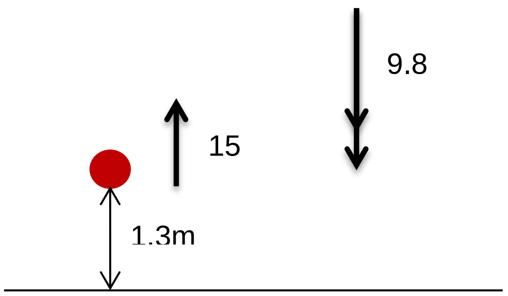
For example, suppose a motorcyclist is travelling in a straight line and decelerates uniformly to rest at a traffic light. If he is initially travelling at 12m/s when he begins to decelerate and it takes him 15 seconds to come to rest, what is his deceleration?
Here, $u = 12, v = 0, t = 15$ and the unknown we are interested in is a. Considering the five $suvat$ equations, the only one which contains a, u, v and t is $v = u + at$

Hence, $0 = 12 + a \times 15$ and rearranging this gives $a = -\frac{4}{5}$

Note that a is negative in this case as the motorcycle is slowing down (decelerating).

Another type of situation to consider involves **vertical motion**. Suppose a person throws a ball vertically upwards from a height of 1.3m with a velocity of 15m/s. How long would it take for the ball to hit the ground? What is the greatest above the ground that the ball will attain?

In this case it is helpful to draw a diagram to ensure the direction of each quantity is measured accurately.



We need to decide which direction to take as positive – vertically upwards or vertically downwards. It doesn't matter which is chosen, as long as it is consistent.

Suppose positive is taken vertically upwards.

Then $u = 15, a = -9.8$ (negative because it is downwards) and when the ball hits the ground it is 1.3m below the starting position, hence $s = -1.3$

The equation which involves u, a, s and t is $s = ut + \frac{1}{2}at^2$ which gives:

$-1.3 = 15t + (\frac{1}{2} \times -9.8 \times t^2)$ or $-1.3 = 15t - 4.9t^2$

Rearranging this quadratic equation to $4.9t^2 - 15t - 1.3 = 0$ and solving using the quadratic formula gives $t = 3.15$ and $t = -0.0843$ to 3s.f. Time cannot be negative and so the time taken to hit the ground is 3.15 seconds.

When thinking about the greatest height above the ground, consider what happens at the maximum point. Here, the ball will have slowed down and is about to start falling back down to the ground. Instantaneously, between the upward and downward parts of the journey, the velocity of the ball will be zero.

Therefore, for the upward part of the journey we have $u = 15, a = -9.8, v = 0$ (instantaneously) and we want to know s. The equation that involves these quantities is $v^2 = u^2 + 2as$ and so $0 = 15^2 + (2 \times -9.8 \times s)$. Hence $s = 11.5$ (3sf). However, the question asks for the maximum height above the ground. 11.5m is the displacement vertically upwards from the starting position and so the distance above the ground is 11.5 + 1.3 = 12.8m.

Task 2

Solve the three problems from earlier:

- The speed camera problem
- The falling book problem
- Playing catch

Solutions

Task 1

$\boldsymbol{v^2 = u^2 + 2as}$

Eliminating t: Rearrange $v = u + at$ to obtain $t = \frac{v-u}{a}$ which can be substituted into the equation $s = \frac{1}{2}(u+v)t$ to get $s = \frac{1}{2}(u+v) \times \left(\frac{v-u}{a}\right)$

Expanding the brackets and simplifying gives $s = \frac{v^2-u^2}{2a}$ which rearranges to give $v^2 = u^2 + 2as$

$\boldsymbol{s = ut + \frac{1}{2}at^2}$

Eliminating v: Substitute the expression $v = u + at$ into $s = \frac{1}{2}(u+v)t$ to obtain $s = \frac{1}{2}(u+u+at)t$ which simplifies to $s = \frac{1}{2}(2u+at)t$ and expanding the brackets gives $s = ut + \frac{1}{2}at^2$

$\boldsymbol{s = vt - \frac{1}{2}at^2}$

Eliminating u: Rearrange $v = u + at$ to obtain $u = v - at$ and substitute this into $s = \frac{1}{2}(u+v)t$ to get $s = \frac{1}{2}(v - at + v)t$ which simplifies to $s = vt - \frac{1}{2}at^2$

Task 2

The speed camera problem: A car accelerates from rest at a set of traffic lights and passes through a speed camera at a distance of 50m from the traffic lights. If the car's acceleration is 1.5 m/s^2, is the car exceeding the speed limit of 40km/h when it passes the camera?

Here, $u = 0, s = 50, a = 1.5$ and we need to know v. Using the equation $v^2 = u^2 + 2as$ we obtain $v^2 = 0 + (2 \times 1.5 \times 50)$ and so $v = 12.2$m/s.

To check if this exceeds 40km/h we need to convert 12.2m/s to km/h by multiplying by 60^2 and dividing by 1000, which gives 43.9km/h. Hence the car is speeding when it passes the camera.

The falling book problem: A book falls from a shelf to the floor, taking 0.45 seconds to land.

How high is the shelf?

Here, the motion is vertically downwards, so it is sensible to take this as the 'positive' direction. Therefore, we know $u = 0, t = 0.45, a = 9.8$ and we need to know s. Using the equation $\boldsymbol{s = ut + \frac{1}{2}at^2}$ gives $s = 0 + \left(\frac{1}{2} \times 9.8 \times 0.45^2\right) = 0.992$ metres.

Playing catch: If a ball is thrown vertically upwards with a speed of 15m/s and is caught in the same position 5 seconds later, find the maximum height of the ball above its starting point.

In a similar way to the earlier example about throwing a ball, in this question there is motion in both the upward and downward directions. If we take upwards as positive, then we have $u = 15, a = -9.8, v = 0$ and we want to know s so we use $v^2 = u^2 + 2as$. This gives $0 = 15^2 + (2 \times -9.8 \times s)$ which gives $s = 11.5$ metres, which is the height above the starting point, as required.