The Further Mathematics Support Programme

Sequences and convergence

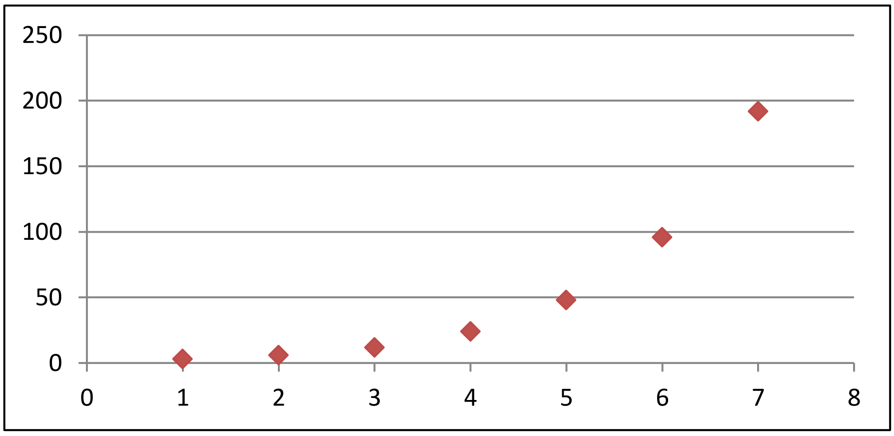
At undergraduate level you will study mathematical analysis, which includes the study of sequences and whether or not they **converge** to a **limit**.

You are likely to be familiar with sequences from previous study of mathematics. For example, if you have studied A level Mathematics you will be aware of **arithmetic sequences** and **geometric sequences**. A geometric sequence increases or decreases from term to term by a common ratio r, starting with a first term a. Let's consider two geometric sequences:

Sequence 1

If $a = 3, r = 2$ the sequence is 3, 6, 12, 24, 48, 96, 192 …..

Here, the terms are getting larger and larger - the sequence is said to be **divergent**. Plotting each term against its position number in the sequence gives the following graph:



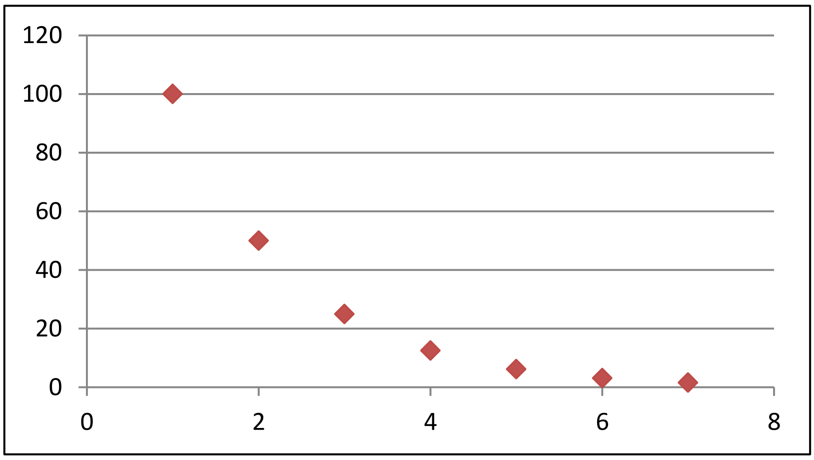
The formula for this sequence is $a_n = 3 \times 2^{n-1}$. The term 2^{n-1} will become increasingly large as n increases. In mathematical notation, we say 'as $n \to \infty, 2^{n-1} \to \infty$' ('as n tends to infinity, 2^{n-1} tends to infinity').

Sequence 2

If $a = 100, r = \frac{1}{2}$ the sequence is 100, 50, 25, 12.5, 6.25, 3.125, 1.5625, ….

Here the terms are getting smaller and smaller towards zero– the sequence is said to be **convergent** and the **limit** of the sequence is 0. Plotting each term against its position number in the sequence gives the graph on the next page.

The formula for this sequence is $a_n = 100 \times \left(\frac{1}{2}\right)^{n-1}$. The term $\frac{1}{2}^{n-1}$ will become increasingly small as n increases, i. e. as $n \to \infty$, $\frac{1}{2}^{n-1} \to 0$



Notice that on the graph for sequence 2, the points get closer and closer to the x-axis, which shows the convergence towards the limit of 0.

So, when $r = 2$ the sequence diverged and when $r = \frac{1}{2}$ the sequence converged. You may recall from A level that a geometric sequence converges if $|r| < 1$, or $-1 < r < 1$.

There are other types of sequences which converge, and some of these can be seen in the task outlined below.

Task

Study each of the following sequences and determine if they are divergent or convergent. If convergent, try to determine the limit of the sequence.

(i) 3, 6, 9, 12, 15, ...

(ii) 4, 4.01, 4.02, 4.03, 4.04, …

(iii) 2000, -1000, 500, -250, 125, …

(iv) 6, $5\frac{1}{2}$, $5\frac{1}{3}$, $5\frac{1}{4}$, $5\frac{1}{5}$, …

(v) 1, 0, 1, 0, 1, …

(vi) 6, 6, 6, 6, 6, …

(vii) 1, 4, 9, 16, 25, …

There is a very formal definition of the limit of a sequence:

Definition:

A real number λ is said to be the **limit** of a sequence of real numbers $\{a_n\}$ if, for every number $\varepsilon > 0$, there exists some integer N such that

$$|a_n - \lambda| < \varepsilon$$

for all $n \geq$ N.

At first sight, this definition seems quite complicated as it is formally stated and uses a lot of mathematical notation. However, the notation actually helps to reduce the number of words that need to be written, and is easy to get used to when used regularly.

$|a_n - \lambda|$ means the absolute value (or modulus) of the difference between term a_n of the sequence and the limit λ. Clearly, as n increases, this difference will become smaller and smaller. At some point, we will get to a position (denoted by N) where the difference is less than a value ε, and that would be true for whatever value of ε we chose. Let's illustrate this with an example:

Example

Suppose $\{a_n\} = \left\{1 + \left(\frac{1}{3}\right)^n\right\}$

Generating the terms of this sequence gives, as improper fractions, $\frac{4}{3}, \frac{10}{9}, \frac{28}{27}, \frac{82}{81}, \frac{244}{243}$, ... and it appears that the limit of the sequence is 1, and that the sequence is converging from above.

Now suppose we choose $\varepsilon = \frac{1}{100}$ and try to find the corresponding value of N. We know:

$|a_n - \lambda| = \left|1 + \left(\frac{1}{3}\right)^n - 1\right| = \left|\left(\frac{1}{3}\right)^n\right| = \left(\frac{1}{3}\right)^n$ and so we need to find a value for which $\left(\frac{1}{3}\right)^n$ is less than ε in order for the definition to be true. The following table will help:

n	1	2	3	4	5	6	7	8	9
$\|a_n - 1\|$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$	$\frac{1}{243}$	$\frac{1}{729}$	$\frac{1}{2187}$	$\frac{1}{6561}$	$\frac{1}{19683}$

So $|a_n - 1| < \frac{1}{100}$ for $n \geq 5$.

If instead we took $\varepsilon = \frac{1}{1000}$ the statement $|a_n - 1| < \varepsilon$ would be true for $n \geq 7$

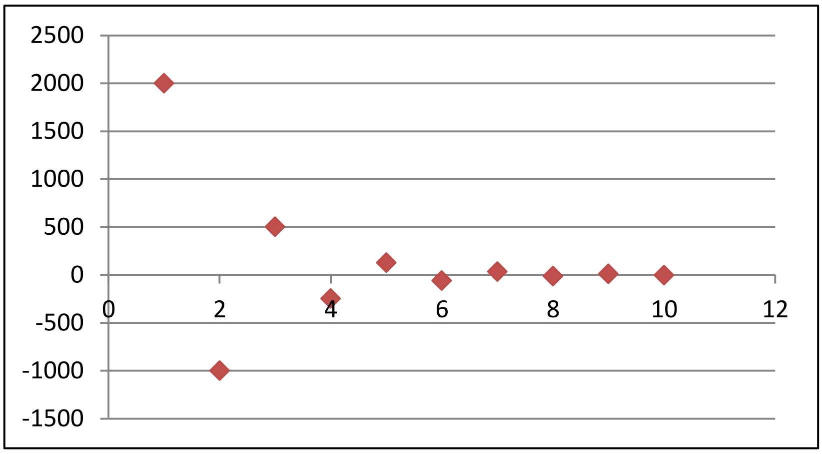
It is in fact possible to prove that, whatever value of ε we choose, $|a_n - \lambda| < \varepsilon$ holds for this sequence, and hence 1 is the limit of the sequence. This is the type of proof that would be part of an undergraduate Analysis module.

Task Solutions

(i) 3, 6, 9, 12, 15, ... is an arithmetic sequence which increases by 3 each time and is divergent as there is no upper limit to the terms.

(ii) 4, 4.01, 4.02, 4.03, 4.04, … is also an arithmetic sequence. Although the terms increase by a very small amount (0.01) each time, they will continue to increase indefinitely and so this sequence is also divergent.

(iii) 2000, -1000, 500, -250, 125, … is a geometric sequences with $a = 2000, r = -\frac{1}{2}$. Here the terms are alternately positive and negative but the absolute value of each term is getting smaller each time. On a graph the terms would look like this:



Here, the terms are said to **oscillate to a limit** and we can see that the limit is 0.

(iv) 6, $5\frac{1}{2}$, $5\frac{1}{3}$, $5\frac{1}{4}$, $5\frac{1}{5}$, … is neither arithmetic nor geometric, but it is convergent. We can see this by first considering the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, ... These terms are getting smaller and smaller and as $n \to \infty$ the sequence $\frac{1}{n}$ converges to 0. (Note that as the denominator gets larger, the fraction gets smaller). Hence, adding 5 on to each term of this sequence means that we still have a converging sequence, but rather than converging to 0, the sequence converges to 5. We say it converges to 5 'from above' as the terms start off above 5 and get closer and closer towards it.

(v) 1, 0, 1, 0, 1, …is an alternating sequence. This does not converge so we say it is divergent.

(vi) 6, 6, 6, 6, 6, … Every term is 6. The sequence is convergent with a limit of 6.

(vii) 1, 4, 9, 16, 25, … This is the sequence of square numbers, with formula $a_n = n^2$. As n increases, $n^2 \to \infty$ and so it is divergent.