The Further Mathematics Support Programme

Simulation

With kind permission, this resource contains some materials from the online **OR-Notes** by J. E. Beasley.

Can you recall the last time you were waiting in a queue? How many people were in front of you in the queue? How many people were serving? Were there separate queues for each server or just one queue that led to all servers?

Queueing situations are one context in which simulation can be used.

Simulation involves the imitation of a real-world situation by using estimations of what might happen in reality. This often involves considering the chance of individual processes occurring, which is called a stochastic simulation.

Task 1

Before you read any further, think about a queueing situation that you are familiar with, for example, queuing in a local fast food venue.

What factors would need to be considered to be able to predict the length of the queue?

Do other factors become more important as the queue gets longer?

In the context you are thinking of, is it better to have one queue to all servers, or separate queues for each server?

Now read on to compare your answers with factors commonly considered in mathematically modelling a queuing situation.

To analyse a queueing situation, we need to consider the things that might vary. For example:

- **arrival process:**
  - how customers arrive e.g. singly or in groups (batch or bulk arrivals)
  - what is the distribution of time between successive arrivals i.e. the **inter-arrival times**

- **service mechanism:**
  - the distribution of how long the service will take i.e. the **service times**
  - the number of servers available
  - whether the servers are in series (each server has a separate queue) or in parallel (one queue for all servers)

- **queue characteristics:**
  - from the set of customers waiting for service, how do we choose the one to be served next? For example, FIFO (first-in first-out); LIFO (last-in first-out); randomly. This is often called the **queue discipline**
  - do we have:
    - balking (customers deciding not to join the queue if it is too long)
    - reneging (customers leave the queue if they have waited too long for service)
    - jockeying (customers switch between queues if they think they will get served faster by so doing)

In the following activities, we will assume a FIFO queue discipline and neglect any factors such as balking or reneging.

When simulating activities it is helpful to use random numbers to generate possible outcomes, such as a person arriving in the queue.

Task 2

1. Identify the random number button on your calculator. It is often denoted by RAN or RAN#. Press this button several times and note the types of numbers that are generated.

2. If you have access to a spreadsheet package such s Excel, use the package to generate twenty random numbers. If using Excel, the command is =RAND().

   How many decimal places are given in each random number?

   Round the numbers to three decimal places by formatting the cells.

We can assume there are 1000 different random numbers, from 0.000 to 0.999. These numbers can be used to generate inter-arrival times and service times, as shown in the following scenario.

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A fast food take away shop wants to reduce queuing times. The manager has recorded the inter-arrival times of customers over a long period of time and has found the following results:

Inter arrival times (secs)	15	30	45	60	75	90
% of occasions	10%	25%	20%	15%	20%	10%

where inter-arrival times are measured to the nearest 15 seconds.

Task 3

Allocate the random numbers 0.000 – 0.999 to the inter-arrival time periods in the same proportions as the percentage of occasions. Fill in the table below with the relevant random number ranges.

For example, 10% of occasions had a 15 second inter-arrival time and so 10% of the random numbers should be allocated to this interval. 10% of 1000 is 10 and so the numbers 0.000 to 0.099 (inclusive) should be allocated to this interval.

Inter-arrival time period (seconds)	% of occasions	How many random numbers?	Random number range
15	10%	100	0.000 – 0.099
30	25%	250	
45	20%	200	
60	15%	150	
75	20%	200	
90	10%	100	

Now generate ten random numbers to represent the first ten customers to arrive at the take-away shop. State the inter-arrival time for each of the customers.

Assuming the shop opens at 12 noon, write down the time of arrival for each customer.

Customer	Random number	Inter-arrival time	Actual time of arrival
1			
2			
3			
4			
5			
6			

7			
8			
9			
10			

Note: A different outcome will be obtained if this experiment is run several times, as the random numbers obtained will be different.

The manager has also recorded the time it takes to serve each individual customer, with the times rounded to the nearest 15 seconds.

Service times (secs)	30	45	60	75	90	105
% of occasions	10%	15%	30%	25%	15%	5%

Task 4

Complete the table below to show the allocation of random numbers to each service time period.

Service time period (seconds)	**% of occasions**	**How many random numbers?**	**Random number range**
15	10%	100	0.000 – 0.099
30	15%	250	
45	30%	200	
60	25%	150	
75	15%	200	
90	5%	100	

Extend your table from Task 3 to include a simulated service time by generating a new set of random numbers. Use this to write down the time of completion of the service for each person. Assuming also that there is only one person serving in the take-away shop and there is a single queue, state the waiting time for each individual.

Customer	**Random number (arrival)**	**Inter-arrival time**	**Actual time of arrival**	**Random number (service)**	**Service duration time**	**Time of completion of service**	**Waiting time**
1							
2							
3							
4							
5							

6							
7							
8							
9							
10							

What was the total customer waiting time in your simulation?

For how long was the server not serving customers?

Does one server seem sufficient, based on your simulation? How could you be more certain?

Task 5

Use the table below to model what would happen if two servers were available. Use the same random numbers and service times as in Task 4. If both servers are free at the same time, assume the customer goes to Server 1. Assume also that all the customers join a single queue.

Customer	Inter-arrival time	Actual time of arrival	Service duration time	Time of completion of service (Server 1)	Time of completion of service (Server 2)	Customer Waiting time
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						

Do you think two servers are needed based on your simulation? What other factors might need to be considered?

Simulation is time consuming and so is often carried out using computer software. Typically a simulation model has to be run on a computer for a considerable time in order for the results to be statistically significant - hence simulations can be expensive (take a long time) in terms of computer time.

Solutions

Task 1

Possible solutions to task 1 are described after the task.

Task 2

1. Random numbers generated on a scientific calculator are usually to 3 decimal places and are between 0 and 1.
2. To save time typing the command into twenty cells, type the command once and 'drag' the grey box at the bottom right hand corner of the cell down to fill twenty rows. The default setting is 9 decimal places but this can be amended by re-formatting the cells.

Task 3

Inter-arrival time period (seconds)	% of occasions	How many random numbers?	Random number range
15	10%	100	0.000 – 0.099
30	25%	250	0.100 – 0.349
45	20%	200	0.350 – 0.549
60	15%	150	0.550 – 0.699
75	20%	200	0.700 – 0.899
90	10%	100	0.900 – 0.999

An **example** of a possible simulation using random numbers is as follows:

Customer	Random number	Inter-arrival time	Actual time of arrival
1	0.725	75	12:01:15
2	0.338	30	12:01:45
3	0.928	90	12:03:15
4	0.161	30	12:03:45
5	0.259	30	12:04:15
6	0.378	45	12:05:00
7	0.169	30	12:05:30

8	0.603	60	12:06:30
9	0.533	45	12:07:15
10	0.601	60	12:08:15

Task 4

Service time period (seconds)	% of occasions	How many random numbers?	Random number range
15	10%	100	0.000 – 0.099
30	15%	150	0.100 – 0.249
45	30%	300	0.250 – 0.549
60	25%	250	0.550 – 0.799
75	15%	150	0.800 – 0.949
90	5%	50	0.950 – 0.999

Continuing the simulation in Example 3 gives:

Customer	Random number (arrival)	Inter-arrival time	Actual time of arrival	Random number (service)	Service duration time	Time of completion of service	Customer Waiting time
1	0.725	75	12:01:15	0.749	60	12:02:15	0
2	0.338	30	12:01:45	0.140	30	12:02:45	30
3	0.928	90	12:03:15	0.043	15	12:03:30	0
4	0.161	30	12:03:45	0.438	45	12:04:30	0
5	0.259	30	12:04:15	0.821	75	12:05:45	15
6	0.378	45	12:05:00	0.571	60	12:06:45	45
7	0.169	30	12:05:30	0.164	30	12:07:15	75
8	0.603	60	12:06:30	0.557	60	12:08:15	45
9	0.533	45	12:07:15	0.276	45	12:09:00	60
10	0.601	60	12:08:15	0.385	45	12:09:45	45

Service of customer 2 starts at 12:02:15 when previous customer leaves.

Customer 3 arrives at 12:03:15, so server has a 30 second gap.

Total customer waiting time is 315 seconds (mean of 31.5 seconds per customer).

Server has gaps of: 30 seconds (between customers 2 and 3) and 15 seconds (between customers 3 and 4) so a total of 45 seconds unoccupied, or a mean of 4.5 seconds per customer (plus the 75 seconds before the first person arrives). In the 9 mins 45 seconds that the shop is open, the server is therefore unoccupied for 20.5% of the time (12.8% of which is waiting for the first customer to arrive).

Based on this simulation, the waiting times seem acceptable, although the waiting times of a minute or more are perhaps too long for a take-away. Running the simulation several times would help us to be more certain of whether the delays are likely to be too long with only one server.

Task 5

If two servers are available:

Customer	Inter-arrival time	Actual time of arrival	Service duration time	Time of completion of service (Server 1)	Time of completion of service (Server 2)	Customer Waiting time
1	75	12:01:15	60	12:02:15	-	0
2	30	12:01:45	30	-	12:02:15	0
3	90	12:03:15	15	12:03:30	-	0
4	30	12:03:45	45	12:04:30	-	0
5	30	12:04:15	75	-	12:05:30	0
6	45	12:05:00	60	12:06:00	-	0
7	30	12:05:30	30	-	12:06:00	0
8	60	12:06:30	60	12:07:30	-	0
9	45	12:07:15	45	-	12:08:00	0
10	60	12:08:15	45	12:09:00	-	0

The customer waiting time is now zero in this simulation, a reduction from 31.5 seconds per customer.

The overall time to complete service of the ten customers is reduced by 45 seconds to 12:09:00, which is only a small saving to say there are now two servers rather than one.

During the 18 minutes of service time (9 per server) since the shop opened, the time spent unoccupied was 56.9% which seems excessive.

However, we have only looked at the first ten customers – a longer simulation might indicate a backlog with only one server, and that in fact two servers will be needed in the longer term. Several repeat simulations are needed to check if this simulation is indicative of the norm.