Degree Topics in Mathematics

Cayley Tables

You will need to look at the *Groups* and *Modular Arithmetic* activities from this set of resources before studying this activity. You will also need a basic understanding of complex numbers for the first example.

A Cayley table is used to display all the different products of a group's elements. This is a useful way to spot patterns, check for commutativity, identify inverse elements and so on. The tables look a bit like possibility spaces used to show the outcomes of two events in probability e.g. the sum of two dice scores.

For example, a Cayley table for the group $\{1, i, -1, -i\}$ under multiplication would look like this:

$*$	**1**	i	**-1**	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Note: Conventionally, the entry at the start of the row is the first element to be considered when multiplying, and the entry at the top of the column comes second. In cases where the binary operation is commutative this is not important, but in other cases it is vital to consider the order in which the operation is carried out.

The table above shows that $\{1, i, -1, -i\}$ under multiplication is a **cyclic** group of **order 4**. Look at the row starting with i. This row contains the first four powers of i and produces all of the elements of the group. The next power of i would be i^5 which equals i, and the pattern continues. i is called a **generator** of the group. Are any of the other elements generators for the group?

Task 1

Using Cayley tables determine whether the following sets are groups under multiplication modulo 3.
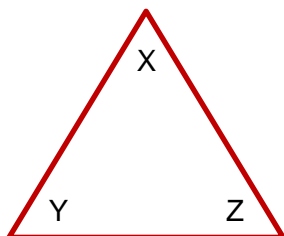
    (i)   {0, 1, 2}   (ii) {1, 2}

Task 2

The table below is a Cayley table.

	A	B	C	D
A	C	A	D	B
B	A	B	C	D
C	D	C	B	A
D	B	D	A	C

(i) Which element is the identity element?

(ii) Which elements are their own inverses (self-inverses)?

(iii) Is this binary operation commutative?

Task 3

Various transformations of this triangle produce a group, which is called a **symmetry group**.

Produce a Cayley table that represents the group (you might want to sketch the triangles as you work out the effect of the combinations of the transformations).

Remember to do the transformation at the start of the row first, followed by the transformation as the top of the column.

Transformations:

e = do nothing

a = rotate 120° anticlockwise

b = rotate 240° anticlockwise

c = reflect in the axis through corner X
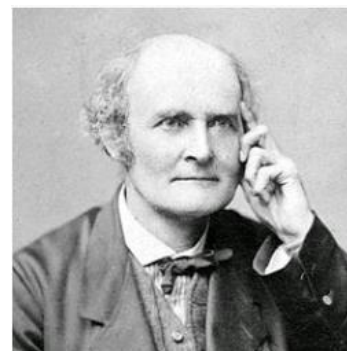
d = reflect in the axis through corner Y

f = reflect in the axis through corner Z

(Think about why 'do nothing' has been labelled e).

	e	a	b	c	d	f
e						
a						
b						
c						
d						
f						

Arthur Cayley was a pure mathematician who was the first to express the concept of a group in the way we use them today. He formulated the Cayley-Hamilton Theorem and had a number of other theorems and formulae named after him, such as Cayley's Mousetrap (a mathematical card game), Cayley's Formula (a result about the kind of graphs used in decision mathematics) and Cayley's sextet (a curve which can be most easily expressed in polar co-ordinates).

Cayley worked as a lawyer for around 14 years, doing maths in his spare time as a hobby. Living in the years 1821-1895, he was an early supporter and advocate of women's participation in higher education, which was not common at the time. (image taken from http://en.wikipedia.org/wiki/Arthur_Cayley)

Solutions

In the example, *-i* is also a generator (try working out the first four or five powers or see the table). 1 and -1 are not considered to be generators as powers of these numbers would not generate all elements of the set.

Task 1

Using Cayley tables determine whether the following sets are groups under multiplication modulo 3.

  (i) {0, 1, 2}   (ii) {1, 2}

(i)

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(ii)

	1	2
1	1	2
2	2	1

In case (i) we see the group is closed and we know that multiplication is associative (this is true for modular multiplication as well as 'normal' multiplication). There is an identity element e=1; however, 0 does not have an inverse – we can see this as there is no 1 in the zero row or column. Hence this is not a group.

In case (ii) however the issue of 0 not having an inverse has been removed. The inverse of 1 is 1, and the inverse of 2 is 2and so this set does form a group.

Task 2

  (i) B is the identity element – multiplying any element by B leaves the element unchanged.

  (ii) An element f has an inverse f^{-1} which satisfies the property $f * f^{-1} = e$. If f is its own inverse then $f * f = e$. Looking at the table in the question, we need an element that produces B (the identity in this case) under binary operation with itself. This is true for elements B and C so they are self-inverses.

  (iii) Checking each pair of elements we can see that the order of multiplication is not important (this can be seen by the symmetry of the table through the leading diagonal). Hence the binary operation is commutative.

Task 3

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e