Applications of A level Mathematics and Further Mathematics

This application makes use of the following:

Topics from A level Mathematics - Calculus – the principle of integration
- Definite integral of a polynomial function
- Mechanics – work done

Topics from GCSE Mathematics - Similar triangles and volume of a cylinder

Work done pumping water from a bore hole

The mechanical work done by a force is the product of the magnitude of the force and the displacement that it moves through.

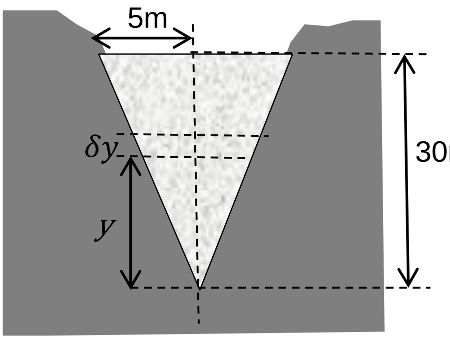
Work = Force x Displacement

In this example the force is the weight of the water in the bore hole which is moved to the surface.

Integration is useful in calculating areas and volumes of shapes. Since both the distance to the surface and weight of water vary with the depth of the hole an understanding of the principles of integration is helpful in setting up and solving a differential equation.

The problem:

How much work is done in pumping water to the surface of a bore hole?



Assumptions:

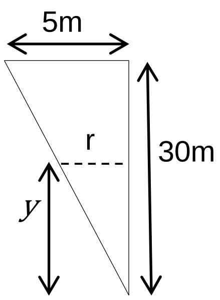
- The water fills a conical hole of depth 30m and base radius 5m.
- Ground level is 30m above the lowest point of the conical hole.
- 1 m^3 of water has a mass of 1000kg.

Setting up the equations:

The work done is equal to the force x distance moved.

The force is equal to the weight of the water.

Start by considering a thin cylindrical disc of water which is a distance y metres from the apex of the cone.



The distance to the surface is (30-y) metres.

Task 1

A: Expression for volume of a disc

Show that the volume of a cylindrical disc of water of thickness δy, a distance y metres from the apex of the cone, is given by $V = \frac{\pi y^2}{36}\delta y$

B: Work done raising a disc to the surface

The weight of water in each disc is raised to the surface.

Find an expression for the work done against gravity in raising the weight of a disc to the surface, in terms of y.

C: Total work done raising the water to the surface

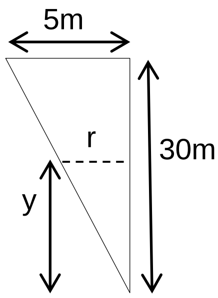
The total work done is the sum of the work done in raising each disc of thickness δy.

As δy tends to zero, then the limit of the sum is the integral.

Show that the total work done is given by $W = -\frac{1000\pi g}{36}\int_0^{30} y^2\,(30 - y)dy$

Hence find W by evaluating this integral.

Solutions



A. Using similar triangles the radius of the cylinder $r = \frac{5y}{30} = \frac{y}{6}$.

Therefore the volume of the cylindrical disc $V = \pi r^2 \delta y$

$$= \pi(\tfrac{y}{6})^2 \delta y = \frac{\pi y^2}{36} \delta y$$

where δy is a very small thickness.

B. The mass of a cylindrical disc is $m = 1000V$ and the weight, $mg = -1000gV$.

The force is negative as the weight is acting down, in the opposite direction to the displacement.

Work done in moving a disc of water to the surface is given by,

$$\text{Work} = -1000gV(30 - y)$$

$$\text{Work} = -1000g\frac{\pi y^2}{36}(30 - y)\,\delta y$$

C. The total work done is the sum of all these thin discs between $y = 0$ and $y = 30$.

As $\delta y \to 0$ this becomes the integral $W = -\frac{1000\pi g}{36}\int_0^{30} y^2\,(30 - y)dy$

This evaluates to $W = -\frac{1000\pi g}{36}\left[10y^3 - \frac{y^4}{4}\right]_0^{30}$

$$W = -\frac{1000\pi g}{36}[270000 - 202500]$$

$$W = -\mathbf{57.73}\ kJ$$

(Note: the work done is negative as it is against gravity.)