KING EDWARD VI SCHOOL

SHAKESPEARE'S SCHOOL

Year 12 A-Level Further Mathematics Practice Booklet

About the Year 12 Mock Examinations

- There will be two 1.5-hour examinations which may can contain content from any of the topics you have studied this year.
- These questions are taken from specimen papers with topics which we have not covered yet removed. Hence the question numbers will not be consecutive.
- You will be allowed a calculator for both papers. Note that AQA's guidance on calculator use is as follows and we shall also apply this in the Year 12 mocks:

> *"If students are asked to "show" or "justify" something, they may need to write more working, but otherwise our attitude will be to expect that students will use whatever calculator functionality they have available."*

- You should take these examinations seriously as they are an important indication of your progress during Year 12. However, they are not the only factor that your teacher will consider when predicting grades for UCAS and your performance across the year and at the beginning of Year 13 will also be considered.

About this Booklet

- These questions are taken from specimen papers with topics which we have not covered yet removed. Hence the question numbers will not be consecutive.
- Every effort has been made to ensure that only questions which have been covered on the Year 12 K.E.S. syllabus have been included. However, there may be some which have slipped through the net. If you are unsure, then please ask your teacher.
- To condense the size of the booklet, the answering space for the questions is not as it would be in an exam. Therefore, if your answer is too long then don't worry; you would have more room in the real thing.

Other Useful Resources

- There are resources on Moodle, including notes interwoven with questions and topic tests from AQA.
- If you want more practice at exam-style questions you can look at past papers from the FP1 – FP4 modules for the old Further Maths A-Level. You will know most, but not all, of the topics on those papers and you should bear this in mind when you encounter something unfamiliar.
- If you have a textbook, then you will also find explanations and additional exercises contained therein.

Topics covered during Year 12

Vectors

- Understand and use the vector and Cartesian forms of an equation of a straight line in 3D.
- Understand and use the vector and Cartesian forms of the equation of a plane.
- Calculate the scalar product and use it to calculate the angle between two lines, to express the equation of a plane, and to calculate the angle between two planes and the angle between a line and a plane.
- Check whether vectors are perpendicular by using the scalar product.
- Calculate and understand the properties of the vector product. Understand and use the equation of a straight line in the form (r – a) × b = 0. Use vector products to find area of a triangle.
- Find the intersection of a line and a line. Find the intersection of a line and a plane. Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane.

Matrices

- Add, subtract and multiply conformable matrices; multiply a matrix by a scalar.
- Understand and use zero and identity matrices.
- Use matrices to represent linear transformations in 2D; successive transformations; single transformations in 3D (3D transformations confined to reflection in one of $x = 0$, $y = 0$, $z = 0$ or rotation about one of the coordinate axes)
- Find invariant points and lines for a linear transformation.
- Calculate determinants of 2 × 2 and 3 × 3 matrices and interpret as scale factors, including the effect on orientation.
- Understand and use singular and non-singular matrices; properties of inverse matrices. Calculate and use the inverse of non-singular 2 × 2 matrices and 3 × 3 matrices.
- Solve three linear simultaneous equations in three variables by use of the inverse matrix.
- Interpret geometrically the solution and failure of solution of three simultaneous linear equations.
- Factorisation of determinants using row and column operations.
- Find eigenvalues and eigenvectors of 2 × 2 and 3 × 3 matrices. Find and use the characteristic equation. Understand the geometrical significance of eigenvalues and eigenvectors.
- Diagonalisation of matrices; $M = UDU^{-1}$; $M^n = UD^nU^{-1}$; when eigenvalues are real.

Complex Numbers

- Solve any quadratic equation with real coefficients; solve cubic or quartic equations with real coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics). Know and use the function e^x and its graph.
- Add, subtract, multiply and divide complex numbers in the form $x + iy$ with x and y real; understand and use the terms 'real part' and 'imaginary part'
- Understand and use the complex conjugate; know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.
- Use and interpret Argand diagrams.

- Convert between the Cartesian form and the modulus-argument form of a complex number.
- Multiply and divide complex numbers in modulus-argument form.
- Construct and interpret simple loci in the Argand diagram such as $|z-a|>r$ and $\arg(z-a)=\theta$.
- Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series.
- Know and use the definition $e^{i\theta}=\cos\theta+i\sin\theta$ and the form $z=re^{i\theta}$
- Find the n distinct n th roots of $re^{i\theta}$ for r ≠ 0 and know that they form the vertices of a regular n -gon in the Argand diagram.
- Use the complex roots of unity to solve geometric problems.

Proof

- Construct proofs using mathematical induction; contexts include sums of series, divisibility, and powers of matrices.

Further Algebra

- Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.
- Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).
- Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.
- Understand and use the method of differences for summation of series including use of partial fractions.
- Find the Maclaurin series of a function including the general term.
- Recognise and use the Maclaurin series for e^x , $\ln(1+x)$, $\sin x$, $\cos x$, and $(1+x)^n$, and be aware of the range of values of x for which they are valid (proof not required).
- Evaluation of limits using Maclaurin series or l'Hôpital's rule.

Further Functions

- Inequalities involving polynomial equations (cubic and quartic).
- Solving inequalities such as $\frac{ax+b}{cx+d} < ex+f$ algebraically.
- Modulus of functions and associated inequalities.
- Graphs of $y=|f(x)|$, $y=\frac{1}{f(x)}$ for given $y=f(x)$
- Graphs of rational functions of form $\frac{ax+b}{cx+d}$; asymptotes, points of intersection with coordinate axes or other straight lines; associated inequalities.
- Graphs of rational functions of form $\frac{ax^2+bx+c}{dx^2+ex+f}$, including cases when some of these coefficients are zero; oblique asymptotes.
- Using quadratic theory (not calculus) to find the possible values of the function and coordinates of the stationary points of the graph for rational functions of form $\frac{ax^2+bx+c}{dx^2+ex+f}$
- Sketching graphs of curves with equations $y^2=4ax$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $xy=c^2$ including intercepts with axes and equations of asymptotes of hyperbolas.

- Single transformations of curves involving translations, stretches parallel to coordinate axes and reflections in the coordinate axes and the lines $y = \pm x$. Extend to composite transformations including rotations and enlargements.

Numerical Methods

- Mid-ordinate rule and Simpson's rule for integration.
- Euler's step by step method for solving first order differential equations.
- Improved Euler method for solving first order differential equations.
  $y_{r+1} = y_{r-1} + 2hf(x_r, y_r) \qquad x_{r+1} = x_r + h$

Hyperbolic Functions

- Understand the definitions of hyperbolic functions $sinh\ x$, $cosh\ x$ and $tanh\ x$, including their domains and ranges, and be able to sketch their graphs.
- Understand the definitions of hyperbolic functions $sech\ x$, $cosech\ x$ and $coth\ x$, including their domains and ranges.
- Differentiate and integrate hyperbolic functions.
- Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.
- Derive and use the logarithmic forms of the inverse hyperbolic functions.
- Integrate functions of the form$(x^2 + a^2)^{-\frac{1}{2}}$ and $(x^2 - a^2)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.
- Understand and use $\tanh x \equiv \frac{\sinh x}{\cosh x}$
- Understand and use $\cosh^2 x - \sinh^2 x \equiv 1$; $\operatorname{sech}^2 x = 1 - \tanh^2 x$ and $\operatorname{cosech}^2 x \equiv \coth^2 x - 1$, $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$, $\sinh 2x \equiv 2 \sinh x \cosh x$
- Construct proofs involving hyperbolic functions and identities.

Polar Coordinates

- Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.
- Sketch curves with r given as a function of θ , including use of trigonometric functions.

Answer ALL questions. Write your answers in the spaces provided.

1. $$f(z) = z^3 + pz^2 + qz - 15$$

where p and q are real constants.

Given that the equation $f(z) = 0$ has roots

$$\alpha, \ \frac{5}{\alpha} \text{ and } \left(\alpha + \frac{5}{\alpha} - 1\right)$$

(a) solve completely the equation $f(z) = 0$

(5)

(b) Hence find the value of p.

(2)

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2. The plane Π passes through the point A and is perpendicular to the vector **n**

Given that

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

where O is the origin,

(a) find a Cartesian equation of Π .

(2)

With respect to the fixed origin O, the line l is given by the equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}$$

The line l intersects the plane Π at the point X.

(b) Show that the acute angle between the plane Π and the line l is 21.2° correct to one decimal place.

(4)

(c) Find the coordinates of the point X.

(4)

3. Tyler invested a total of £5000 across three different accounts; a savings account, a property bond account and a share dealing account.

Tyler invested £400 more in the property bond account than in the savings account.

After one year

- the savings account had increased in value by 1.5%
- the property bond account had increased in value by 3.5%
- the share dealing account had **decreased** in value by 2.5%

- the total value across Tyler's three accounts had increased by £79

Form and solve a matrix equation to find out how much money was invested by Tyler in each account.

(7)

4. The cubic equation

$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

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5. $$\mathbf{M} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

(a) Show that **M** is non-singular.
(2)

The hexagon R is transformed to the hexagon S by the transformation represented by the matrix **M**.

Given that the area of hexagon R is 5 square units,

(b) find the area of hexagon S.
(1)

The matrix **M** represents an enlargement, with centre (0, 0) and scale factor k, where $k > 0$, followed by a rotation anti-clockwise through an angle θ about (0, 0).

(c) Find the value of k.
(2)

(d) Find the value of θ.
(2)

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6. (a) Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of n that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

8. (a) Shade on an Argand diagram the set of points

$$\left\{z \in \mathbb{C} : |z - 4\mathrm{i}| \leqslant 3\right\} \cap \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z + 3 - 4\mathrm{i}) \leqslant \frac{\pi}{4}\right\}$$

(6)

The complex number w satisfies

$$|w - 4\mathrm{i}| = 3$$

(b) Find the maximum value of $\arg w$ in the interval $(-\pi, \pi]$.
Give your answer in radians correct to 2 decimal places.

(2)

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9. An octopus is able to catch any fish that swim within a distance of 2 m from the octopus's position.

A fish F swims from a point A to a point B.

The octopus is modelled as a fixed particle at the origin O.

Fish F is modelled as a particle moving in a straight line from A to B.

Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish F.

(7)

(b) Criticise the model in relation to fish F.

(1)

(c) Criticise the model in relation to the octopus.

(1)

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Paper 1: Core Pure Mathematics Mark Scheme

Question	Scheme	Marks	AOs
1(a)	$\alpha\left(\frac{5}{\alpha}\right)\left(\alpha+\frac{5}{\alpha}-1\right)=15$	M1	1.1b
		A1	1.1b
	$\Rightarrow 5\alpha+\frac{25}{\alpha}-5=15 \Rightarrow \alpha^2-4\alpha+5=0$ $\Rightarrow \alpha=\frac{--4\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$ or $(\alpha-2)^2-4+5=0 \Rightarrow \alpha=...$	M1	3.1a
	$\Rightarrow \alpha = 2 \pm \text{i}$	A1	1.1b
	Hence the roots of $\text{f}(z)=0$ are $2+\text{i}, 2-\text{i}$ and 3	A1	2.2a
		(5)	
(b)	$p=-\left(\text{"}(2+\text{i})\text{"}+\text{"}(2-\text{i})\text{"}+\text{"}3\text{"}\right) \Rightarrow p=...$	M1	3.1a
	$\Rightarrow p=-7$ cso	A1	1.1b
		(2)	
	1(b) alternative		
	$\text{f}(z)=(z-3)(z^2-4z+5) \Rightarrow p=...$	M1	3.1a
	$\Rightarrow p=-7$ cso	A1	1.1b
		(2)	
			(7 marks)

Notes:

(a)

M1: Multiplies the three given roots together and sets the result equal to 15 or -15

A1: Obtains a correct equation in α

M1: Forms a quadratic equation in α and attempts to solve this equation by either completing the square or using the quadratic formula to give $\alpha = ...$

A1: $\alpha = 2 \pm \text{i}$

A1: Deduces the roots are $2+\text{i}, 2-\text{i}$ and 3

(b)

M1: Applies the process of finding $-\sum\left(\text{of their three roots found in part } (a)\right)$ to give $p = ...$

A1: $p=-7$ by correct solution only

(b) Alternative

M1: Applies the process expanding $(z-\text{"}3\text{"})(z-(\text{their sum})z+\text{their product})$ in order to find $p = ...$

A1: $p=-7$ by correct solution only

Question	Scheme	Marks	AOs
2(a)	$\mathbf{r}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}5\\-3\\-4\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\2\end{pmatrix}$	M1	1.1b
	$3x-y+2z=10$	A1	2.5
		(2)	
(b)	$\begin{pmatrix}3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\-5\\3\end{pmatrix}=8$	B1	1.1b
	$\sqrt{(3)^2+(-1)^2+(2)^2}.\sqrt{(-1)^2+(-5)^2+(3)^2}\cos\alpha = "-3+5+6"$	M1	1.1b
	$\theta=90^\circ-\arccos\left(\frac{8}{\sqrt{14}.\sqrt{35}}\right)$ or $\sin\theta=\frac{8}{\sqrt{14}.\sqrt{35}}$	M1	2.1
	$\theta=21.2°$ (1 dp) * cso	A1*	1.1b
		(4)	
(c)	$3(7-\lambda)-(3-5\lambda)+2(-2+3\lambda)=10 \Rightarrow \lambda=\ldots$	M1	3.1a
	$\lambda=-\frac{1}{2}$	A1	1.1b
	$\overrightarrow{OX}=\begin{pmatrix}7\\3\\-2\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\-5\\3\end{pmatrix}=\begin{pmatrix}\ldots\\\ldots\\\ldots\end{pmatrix}$	M1	1.1b
	$X(7.5, 5.5, -3.5)$	A1ft	1.1b
		(4)	
		(10 marks)	

Notes:

(a)

M1: Attempts to apply the formula $\mathbf{r.n}=\mathbf{a.n}$

A1: Correct Cartesian notation. e.g. $3x-y+2z=10$ or $-3x+y-2z=-10$

Note: Do not allow final answer given as $\mathbf{r}\bullet(3\mathbf{i}-\mathbf{j}+2\mathbf{k})=10$, o.e.

(b)

B1: $\overline{OA}\bullet\mathbf{n}=8$

M1: An attempt to apply the correct dot product formula between **n** and **d**

M1: Depends on previous M mark. Applies the dot product formula to find the angle between Π and l

A1*: 21.2° cso

Question 2 notes continued:	
(c)	
M1:	Substitutes l into Π and solves the resulting equation to give $\lambda = ...$
A1:	$\lambda = -\frac{1}{2}$ o.e.
M1:	Depends on previous M mark. Substitutes their λ into l and finds at least one of the coordinates
A1ft:	$(7.5, 5.5, -3.5)$ but follow through on their value of λ

Question	Scheme	Marks	AOs
3	x = value of savings account, y = value of property bond account, z = value of share dealing account	M1	3.1b
	$x+y+z=5000$ $x+400=y$ $0.015x+0.035y-0.025z=79$ **or** $1.015x+1.035y+0.975z=5079$	A1	1.1b
	Let $\mathbf{A}=\begin{pmatrix}1&1&1\\1&-1&0\\0.015&0.035&-0.025\end{pmatrix}$ or $\begin{pmatrix}1&1&1\\1&-1&0\\1.015&1.035&0.975\end{pmatrix}$		
	e.g. $\begin{pmatrix}1&1&1\\1&-1&0\\0.015&0.035&-0.025\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5000\\-400\\79\end{pmatrix}$	M1	3.1a
		A1	1.1b
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&1&1\\1&-1&0\\0.015&0.035&-0.025\end{pmatrix}^{-1}\begin{pmatrix}5000\\-400\\79\end{pmatrix}=\begin{pmatrix}...\\...\\...\end{pmatrix}$	M1	1.1b
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1800\\2200\\1000\end{pmatrix}$	A1	1.1b
	Tyler invested £1800 in the savings account, £2200 in the property bond account and £1000 in the share dealing account	A1ft	3.2a
			(7 marks)

Notes:

M1: Attempts to set up 3 equations with 3 unknowns

A1: At least 2 equations are correct with the appropriate variables defined

M1: Sets up a matrix equation of the form, e.g. $\begin{pmatrix}...&...&...\\...&...&...\\...&...&...\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}...\\...\\...\end{pmatrix}$, where "…" are numerical values

A1: Correct matrix equation (or equivalent)

M1: Depends on previous M mark. Applies $(\text{their } \mathbf{A})^{-1}\begin{pmatrix}5000\\\text{their "}-400\text{"}\\\text{their "79"}\end{pmatrix}$ and obtains at least one value of x, y or z

A1: Correct answer

A1ft: Correct follow through answer in context

Question	Scheme	Marks	AOs
4	$\{w = x-1 \Rightarrow\}\ x = w+1$	B1	3.1a
	$(w+1)^3 + 3(w+1)^2 - 8(w+1) + 6 = 0$	M1	3.1a
	$w^3 + 3w^2 + 3w + 1 + 3(w^2 + 2w + 1) - 8w - 8 + 6 = 0$		
	$w^3 + 6w^2 + w + 2 = 0$	M1	1.1b
		A1	1.1b
		A1	1.1b
		(5)	
	Alternative		
	$\alpha + \beta + \gamma = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -8,\ \alpha\beta\gamma = -6$	B1	3.1a
	sum roots $= \alpha - 1 + \beta - 1 + \gamma - 1$	M1	3.1a
	$= \alpha + \beta + \gamma - 3 = -3 - 3 = -6$		
	pair sum $= (\alpha-1)(\beta-1) + (\alpha-1)(\gamma-1) + (\beta-1)(\gamma-1)$		
	$= \alpha\beta + \alpha\gamma + \beta\gamma - 2(\alpha+\beta+\gamma) + 3$		
	$= -8 - 2(-3) + 3 = 1$		
	product $= (\alpha - 1)(\beta - 1)(\gamma - 1)$		
	$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha+\beta+\gamma) - 1$		
	$= -6 - (-8) - 3 - 1 = -2$		
	$w^3 + 6w^2 + w + 2 = 0$	M1	1.1b
		A1	1.1b
		A1	1.1b
		(5)	
	(5 marks)		

Notes:

B1: Selects the method of making a connection between x and w by writing $x = w+1$

M1: Applies the process of substituting their $x = w+1$ into $x^3 + 3x^2 - 8x + 6 = 0$

M1: Depends on previous M mark. Manipulating their equation into the form $w^3 + pw^2 + qw + r = 0$

A1: At least two of p, q, r are correct

A1: Correct final equation

Alternative

B1: Selects the method of giving three correct equations each containing α, β and γ

M1: Applies the process of finding sum roots, pair sum and product

M1: Depends on previous M mark. Applies $w^3 - (\text{their sum roots})w^2 + (\text{their pair sum})w - \text{their } \alpha\beta\gamma = 0$

A1: At least two of p, q, r are correct

A1: Correct final equation

Question	Scheme	Marks	AOs
5(a)	$\det(\mathbf{M}) = (1)(1) - (\sqrt{3})(-\sqrt{3})$	M1	1.1a
	M is non-singular because $\det(\mathbf{M}) = 4$ and so $\det(\mathbf{M}) \neq 0$	A1	2.4
		(2)	
(b)	$\text{Area}(S) = 4(5) = 20$	B1ft	1.2
		(1)	
(c)	$k = \sqrt{(1)(1) - (\sqrt{3})(-\sqrt{3})}$	M1	1.1b
	$= 2$	A1ft	1.1b
		(2)	
(d)	$\cos\theta = \frac{1}{2}$ or $\sin\theta = \frac{\sqrt{3}}{2}$ or $\tan\theta = \sqrt{3}$	M1	1.1b
	$\theta = 60°$ or $\frac{\pi}{3}$	A1	1.1b
		(2)	
			(7 marks)

Notes:

(a)

M1: An attempt to find $\det(\mathbf{M})$.

A1: $\det(\mathbf{M}) = 4$ **and** reference to zero, e.g. $4 \neq 0$ **and** conclusion.

(b)

B1ft: 20 or a correct ft based on their answer to part (a).

(c)

M1: $\sqrt{(\text{their } \det\mathbf{M})}$

A1ft: 2

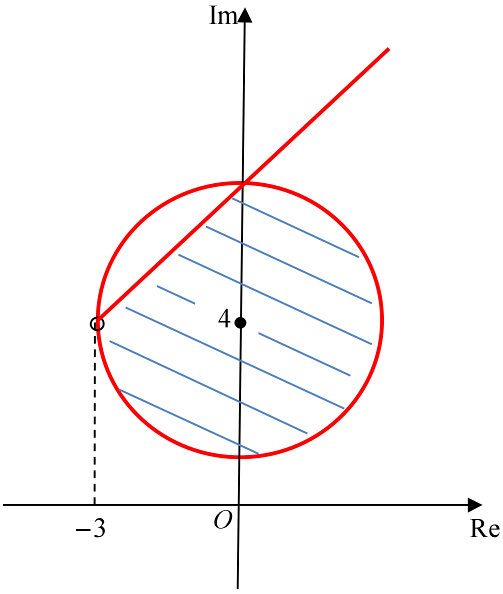
(d)

M1: **Either** $\cos\theta = \frac{1}{(\text{their } k)}$ **or** $\sin\theta = \frac{\sqrt{3}}{(\text{their } k)}$ **or** $\tan\theta = \sqrt{3}$

A1: $\theta = 60^\circ$ or $\frac{\pi}{3}$. Also accept any value satisfying $360n + 60^\circ$, $n \in \mathbb{Z}$, o.e.

Question	Scheme	Marks	AOs
6(a)	$n=1, \sum_{r=1}^{1} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(1)(2)(3) = 1$	B1	2.2a
	Assume general statement is true for $n=k$ So assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true	M1	2.4
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$	M1	2.1
	$= \frac{1}{6}(k+1)(2k^2+7k+6)$	A1	1.1b
	$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$	A1	1.1b
	Then the general result is <u>true for $n=k+1$</u> As the general result has been shown to be <u>true for $n=1$,</u> then the general result <u>is true for all n</u> $\in \mathbb{Z}^+$	A1	2.4
		(6)	
(b)	$\sum_{r=1}^{n} r(r+6)(r-6) = \sum_{r=1}^{n} (r^3 - 36r)$		
	$= \frac{1}{4}n^2(n+1)^2 - \frac{36}{2}n(n+1)$	M1 A1	2.1 1.1b
	$= \frac{1}{4}n(n+1)\left[n(n+1) - 72\right]$	M1	1.1b
	$= \frac{1}{4}n(n+1)(n-8)(n+9)$ * cso	A1*	1.1b
		(4)	
(c)	$\frac{1}{4}n(n+1)(n-8)(n+9) = \frac{17}{6}n(n+1)(2n+1)$	M1	1.1b
	$\frac{1}{4}(n-8)(n+9) = \frac{17}{6}(2n+1)$	M1	1.1b
	$3n^2 - 65n - 250 = 0$	A1	1.1b
	$(3n+10)(n-25) = 0$	M1	1.1b
	(As n must be a positive integer,) $n = 25$	A1	2.3
		(5)	
			(15 marks)

Question 6 notes:
(a) **B1:** Checks $n=1$ works for both sides of the general statement **M1:** Assumes (general result) true for $n=k$ **M1:** Attempts to add $(k+1)^{\text{th}}$ term to the sum of k terms **A1:** Correct algebraic work leading to **either** $\frac{1}{6}(k+1)(2k^2+7k+6)$ **or** $\frac{1}{6}(k+2)(2k^2+5k+3)$ **or** $\frac{1}{6}(2k+3)(k^2+3k+2)$ **A1:** Correct algebraic work leading to $\frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$ **A1:** cso leading to a correct induction statement conveying all three underlined points
(b) **M1:** Substitutes at least one of the standard formulae into their expanded expression **A1:** Correct expression **M1:** Depends on previous M mark. Attempt to factorise at least $n(n+1)$ having used **A1*:** Obtains $\frac{1}{4}n(n+1)(n-8)(n+9)$ by cso
(c) **M1:** Sets their part (a) answer equal to $\frac{17}{6}n(n+1)(2n+1)$ **M1:** Cancels out $n(n+1)$ from both sides of their equation **A1:** $3n^2-65n-250=0$ **M1:** A valid method for solving a 3 term quadratic equation **A1:** Only one solution of $n=25$

Question	Scheme	Marks	AOs
8(a)		M1 A1 M1 A1 M1 A1	1.1b 1.1b 1.1b 2.2a 3.1a 1.1b
		(6)	
(b)	$(\arg w)_{\max} = \frac{\pi}{2} + \arcsin\left(\frac{3}{4}\right)$	M1	3.1a
	$= 2.42$ (2dp) **cao**	A1	1.1b
		(2)	
		(8 marks)	

Notes:

(a)
M1: Circle
A1: Centre (0, 4) and above the real axis
M1: Half-line
A1: $(-3, 4)$ positioned correctly and the half-line intersects the top of the circle on the y-axis
M1: Depends on **both** previous M marks Shades in a region inside the circle and below the half-line
A1: cso

Note: Final A1 mark is dependent on all previous marks being scored in part (a)

(b)
M1: Uses trigonometry to give an expression for an angle in the range $\left(\frac{\pi}{2}, \pi\right)$ or $(90°, 180°)$
A1: 2.42 cao

Question	Scheme	Marks	AOs
9(a)	$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ 11 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \left\{ = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} \right\}$ or $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$	M1	3.1a
	$\left\{ \overrightarrow{OF} = \mathbf{r} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$	M1	1.1b
	$\left\{ \overrightarrow{OF} \bullet \overrightarrow{AB} = 0 \Rightarrow \right\} \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix} \bullet \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = 0$ $\Rightarrow -36 + 144\lambda + 3 + 9\lambda - 126 + 324\lambda = 0 \Rightarrow 477\lambda - 159 = 0$	dM1	1.1b
	$\Rightarrow \lambda = \frac{1}{3}$	A1	1.1b
	$\left\{ \overrightarrow{OF} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and minimum distance $= \sqrt{(1)^2 + (2)^2 + (-1)^2}$	dM1	3.1a
	$= \sqrt{6}$ **or** $2.449\ldots$	A1	1.1b
	> 2, so the octopus is not able to catch the fish F	A1ft	3.2a
		(7)	

Question	Scheme	Marks	
	9(a) Alternative 1		
	$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ 11 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \left\{= \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}\right\}$ or $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$	M1	3.1a
	$\left\{\overrightarrow{OA} = \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \text{ and } \overrightarrow{AB} = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} \Rightarrow\right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$	M1	1.1b
	$\cos\theta \left\{= \dfrac{\overrightarrow{OA} \bullet \overrightarrow{AB}}{\lvert\overrightarrow{OA}\rvert . \lvert\overrightarrow{AB}\rvert}\right\} = \dfrac{\pm\left(\begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}\right)}{\sqrt{(-3)^2 + (1)^2 + (-7)^2} . \sqrt{(12)^2 + (3)^2 + (18)^2}}$	dM1	1.1b
	$\left\{\cos\theta = \dfrac{-36 + 3 - 126}{\sqrt{59}.\sqrt{477}} = \dfrac{-159}{\sqrt{59}.\sqrt{477}}\right\}$		
	$\theta = 161.4038029...$ or $18.59619709...$ or $\sin\theta = 0.3188964021...$	A1	1.1b
	minimum distance $= \sqrt{(-3)^2 + (1)^2 + (-7)^2}\sin(18.59619709...)$	dM1	3.1a
	$= \sqrt{6}$ or $2.449\ldots$	A1	1.1b
	> 2, so the octopus is not able to catch the fish F	A1ft	3.2a
		(7)	
	9(a) Alternative 2		
	$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ 11 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \left\{= \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}\right\}$ or $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$	M1	3.1a
	$\left\{\overrightarrow{OF} = \mathbf{r} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$	M1	1.1b
	$\lvert\overrightarrow{OF}\rvert^2 = (-3 + 12\lambda)^2 + (1 + 3\lambda)^2 + (-7 + 18\lambda)^2$	dM1	1.1b
	$= 9 - 72\lambda + 144\lambda^2 + 1 + 6\lambda + 9\lambda^2 + 49 - 252\lambda + 324\lambda^2$		
	$= 477\lambda^2 - 318\lambda + 59$	A1	1.1b
	$= 53(3\lambda - 1)^2 + 6$	dM1	3.1a
	minimum distance $= \sqrt{6}$ or $2.449\ldots$	A1	1.1b
	> 2, so the octopus is not able to catch the fish F	A1ft	3.2a
		(7)	

Question	Scheme	Marks	AOs
9(b)	e.g. Fish F may not swim in an exact straight line from A to B Fish F may hit an obstacle whilst swimming from A to B Fish F may deviate his path to avoid being caught by the octopus	B1	3.5b
		(1)	
(c)	e.g. Octopus is effectively modelled as a particle – so we may need to look at where the octopus's mass is distributed Octopus may during the fish F's motion move away from its fixed location at O	B1	3.5b
		(1)	
			(9 marks)

Question 9 notes:

(a)

M1: Attempts to find $\overrightarrow{OB} - \overrightarrow{OA}$ **or** $\overrightarrow{OA} - \overrightarrow{OB}$ **or** the direction vector **d**

M1: Applies $\overrightarrow{OA} + \lambda(\text{their } \overrightarrow{AB} \text{ or their } \overrightarrow{BA} \text{ or their } \mathbf{d})$ or equivalent

M1: Depends on previous M mark. Writes down (their $\overrightarrow{OF}$ which is in terms of λ)•(their $\overrightarrow{AB}$) = 0. Can be implied

A1: Lambda is correct. e.g. $\lambda = \frac{1}{3}$ for $\overrightarrow{AB} = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$ or $\lambda = 1$ for $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$

M1: Depends on previous M mark. Complete method for finding $|\overrightarrow{OF}|$

A1: $\sqrt{6}$ **or** awrt 2.4

A1ft : Correct follow through conclusion, which is in context with the question

Alternative 1

(a)

M1: Attempts to find $\overrightarrow{OB} - \overrightarrow{OA}$ **or** $\overrightarrow{OA} - \overrightarrow{OB}$ **or** the direction vector **d**

M1: Realisation that the dot product is required between $\overrightarrow{OA}$ and their $\overrightarrow{AB}$. (o.e.)

M1: Depends on previous M mark. Applies dot product formula between $\overrightarrow{OA}$ and their $\overrightarrow{AB}$ (o.e.)

A1: θ = awrt 161.4 or awrt 18.6 or $\sin\theta$ = awrt 0.319

M1: Depends on previous M mark. (their OA)sin(their θ)

A1: $\sqrt{6}$ or awrt 2.4

A1ft : Correct follow through conclusion, which is in context with the question

Question 9 notes continued:
Alternative 2 **(a)** **M1:** Attempts to find $\overrightarrow{OB} - \overrightarrow{OA}$ **or** $\overrightarrow{OA} - \overrightarrow{OB}$ **or** the direction vector **d** **M1:** Applies $\overrightarrow{OA} + \lambda(\text{their } \overrightarrow{AB} \text{ or their } \overrightarrow{BA} \text{ or their } \mathbf{d})$ or equivalent **M1:** Depends on previous M mark. Applies Pythagoras by finding $\left\|\overrightarrow{OF}\right\|^2$, o.e. **A1:** $\left\|\overrightarrow{OF}\right\|^2 = 477\lambda^2 - 318\lambda + 59$ **M1:** Depends on previous M mark. Method of completing the square or differentiating their $\left\|\overrightarrow{OF}\right\|^2$ w.r.t. λ **A1:** $\sqrt{6}$ or awrt 2.4 **A1ft :** Correct follow through conclusion, which is in context with the question
(b) **B1:** An acceptable criticism for fish F, which is in context with the question
(c) **B1:** An acceptable criticism for the octopus, which is in context with the question

2. The value, V hundred pounds, of a particular stock t hours after the opening of trading on a given day is modelled by the differential equation

$$\frac{dV}{dt} = \frac{V^2 - t}{t^2 + tV} \qquad 0 < t < 8.5$$

A trader purchases £300 of the stock one hour after the opening of trading.

Use two iterations of Euler's formula for approximating differential equations to estimate, to the nearest £, the value of the trader's stock half an hour after it was purchased.

(6)

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3. Use algebra to find the set of values of x for which

$$\frac{1}{x} < \frac{x}{x+2}$$

(6)

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5.

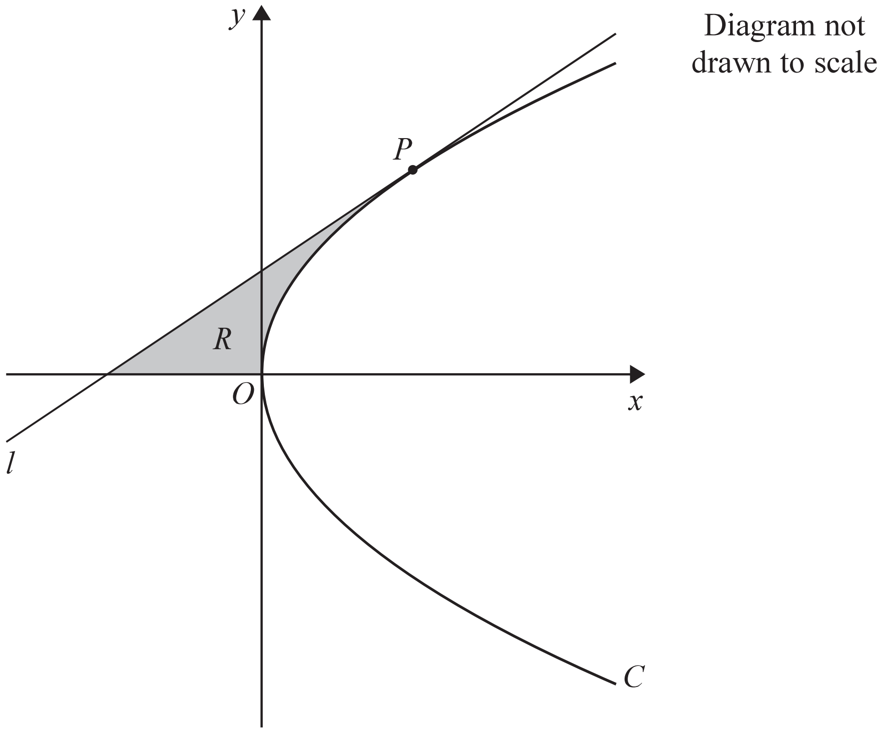

Figure 2

[You may quote without proof that for the general parabola $y^2 = 4ax$, $\frac{dy}{dx} = \frac{2a}{y}$]

The parabola C has equation $y^2 = 16x$.

(a) Deduce that the point $P(4p^2, 8p)$ is a general point on C.

(1)

The line l is the tangent to C at the point P.

(b) Show that an equation for l is

$$py = x + 4p^2$$

(3)

The finite region R, shown shaded in Figure 2, is bounded by the line l, the x-axis and the parabola C.

The line l intersects the directrix of C at the point B, where the y coordinate of B is $\frac{10}{3}$

Given that $p > 0$

(c) show that the area of R is 36

(8)

Question 5 continued

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SECTION B

Answer ALL questions. Write your answers in the spaces provided.

6. Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$$

(a) find the characteristic equation of the matrix **A**.

(2)

(b) Hence show that $\mathbf{A}^3 = 43\mathbf{A} - 42\mathbf{I}$.

(3)

8. A curve C is described by the equation

$$|z - 9 + 12\mathrm{i}| = 2|z|$$

(a) Show that C is a circle, and find its centre and radius.

(4)

(b) Sketch C on an Argand diagram.

(2)

Given that w lies on C,

(c) find the largest value of a and the smallest value of b that must satisfy

$$a \leqslant \mathrm{Re}(w) \leqslant b$$

(2)

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10. A population of deer on a large estate is assumed to increase by 10% during each year due to natural causes.

The population is controlled by removing a constant number, Q, of the deer from the estate at the end of each year.

At the start of the first year there are 5000 deer on the estate.

Let P_n be the population of deer at the end of year n.

(a) Explain, in the context of the problem, the reason that the deer population is modelled by the recurrence relation

$$P_n = 1.1P_{n-1} - Q, \quad P_0 = 5000, \quad n \in \mathbb{Z}^+$$

(3)

(b) Prove by induction that $P_n = (1.1)^n (5000 - 10Q) + 10Q, \quad n \geqslant 0$

(5)

(c) Explain how the long term behaviour of this population varies for different values of Q.

(2)

Question 10 continued

Question 10 continued

Question 10 continued

(Total for Question 10 is 10 marks)

TOTAL FOR SECTION B IS 40 MARKS
TOTAL FOR PAPER IS 80 MARKS

Question	Scheme	Marks	AOs
2	£300 purchased one hour after opening $\Rightarrow V_0 = 3$ and $t_0 = 1$; half an hour after purchase $\Rightarrow t_2 = 1.5$, so step h required is 0.25	B1	3.3
	$t_0 = 1,\ V_0 = 3,\ \left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_0 \approx \frac{3^2 - 1}{1^2 + 3} = 2$	M1	3.4
	$V_1 \approx V_0 + h\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_0 = 3 + 0.25 \times 2 = \ldots$	M1	1.1b
	$= 3.5$	A1ft	1.1b
	$\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_1 \approx \frac{3.5^2 - 1.25}{1.25^2 + 1.25 \times 3.5}\ \left(= \frac{176}{95}\right)$	M1	1.1b
	$V_2 \approx V_1 + h\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_1 = 3.5 + 0.25 \times \frac{176}{95} = 3.963\ldots$, so £396 (nearest £)	A1	3.2a
		(6)	
		(6 marks)	

Notes:

B1: Identifies the correct initial conditions and requirement for h

M1: Uses the model to evaluate $\frac{\mathrm{d}V}{\mathrm{d}t}$ at t_0, using their t_0 and V_0

M1: Applies the approximation formula with their values

A1ft: 3.5 or exact equivalent. Follow through their step value

M1: Attempt to find $\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)_1$ with their 3.5

A1: Applies the approximation and interprets the result to give £396

Question	Scheme	Marks	AOs
3	$\frac{1}{x} < \frac{x}{x+2}$		
	$\frac{(x+2)-x^2}{x(x+2)} < 0$ or $x(x+2)^2 - x^3(x+2) < 0$	M1	2.1
	$\frac{x^2-x-2}{x(x+2)} > 0 \Rightarrow \frac{(x-2)(x+1)}{x(x+2)} > 0$ or $x(x+2)(2-x)(x+1) < 0$	M1	1.1b
	At least two correct critical values from $-2, -1, 0, 2$	A1	1.1b
	All four correct critical values $-2, -1, 0, 2$	A1	1.1b
	$\{x \in \mathbb{R} : x < -2\} \cup \{x \in \mathbb{R} : -1 < x < 0\} \cup \{x \in \mathbb{R} : x > 2\}$	M1 A1	2.2a 2.5
		(6)	
		(6 marks)	

Notes:

M1: Gathers terms on one side and puts over common denominator, or multiply by $x^2(x+2)^2$ and then gather terms on one side

M1: Factorise numerator or find roots of numerator or factorise resulting in equation into 4 factors

A1: At least 2 correct critical values found

A1: Exactly 4 correct critical values

M1: Deduces that the 2 "outsides" and the "middle interval" are required. May be by sketch, number line or any other means

A1: Exactly 3 correct intervals, accept equivalent set notations, but must be given as a set e.g. accept $\mathbb{R} - ([-2, -1] \cup [0, 2])$ or $\{x \in \mathbb{R} : x < -2 \text{ or } -1 < x < 0 \text{ or } x > 2\}$

Question	Scheme	Marks	AOs
5(a)	$y^2 = (8p)^2 = 64p^2$ and $16x = 16(4p^2) = 64p^2$ $\Rightarrow P(4p^2, 8p)$ is a general point on C	B1	2.2a
		(1)	
(b)	$y^2 = 16x$ gives $a = 4$, or $2y\dfrac{dy}{dx} = 16$ so $\dfrac{dy}{dx} = \dfrac{8}{y}$	M1	2.2a
	$l: y - 8p = \left(\dfrac{8}{8p}\right)\left(x - 4p^2\right)$	M1	1.1b
	leading to $py = x + 4p^2$ *	A1*	2.1
		(3)	
(c)	$B\left(-4, \dfrac{10}{3}\right)$ into $l \Rightarrow \dfrac{10p}{3} = -4 + 4p^2$	M1	3.1a
	$6p^2 - 5p - 6 = 0 \Rightarrow (2p-3)(3p+2) = 0 \Rightarrow p = ...$	M1	1.1b
	$p = \dfrac{3}{2}$ and l cuts x-axis when $\dfrac{3}{2}(0) = x + 4\left(\dfrac{3}{2}\right)^2 \Rightarrow x = ...$	M1	2.1
	$x = -9$	A1	1.1b
	$p = \dfrac{3}{2} \Rightarrow P(9, 12) \Rightarrow \text{Area}(R) = \dfrac{1}{2}(9 - -9)(12) - \int_0^9 4x^{\frac{1}{2}}dx$	M1	2.1
	$\int 4x^{\frac{1}{2}}dx = \dfrac{4x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} (+c)$ or $\dfrac{8}{3}x^{\frac{3}{2}} (+c)$	M1	1.1b
		A1	1.1b
	$\text{Area}(R) = \dfrac{1}{2}(18)(12) - \dfrac{8}{3}\left(9^{\frac{3}{2}} - 0\right) = 108 - 72 = 36$ *	A1*	1.1b
		(8)	

Question	Scheme	Marks	AOs
	5(c) Alternative 1		
	$B\left(-4, \frac{10}{3}\right)$ into $l \Rightarrow \frac{10p}{3} = -4 + 4p^2$	M1	3.1a
	$6p^2 - 5p - 6 = 0 \Rightarrow (2p-3)(3p+2) = 0 \Rightarrow p = ...$	M1	1.1b
	$p = \frac{3}{2}$ into l gives $\frac{3}{2}y = x + 4\left(\frac{3}{2}\right)^2 \Rightarrow x = ...$	M1	2.1
	$x = \frac{3}{2}y - 9$	A1	1.1b
	$p = \frac{3}{2} \Rightarrow P(9, 12) \Rightarrow \text{Area}(R) = \int_0^{12}\left(\frac{1}{16}y^2 - \left(\frac{3}{2}y - 9\right)\right)\mathrm{d}y$	M1	2.1
	$\int\left(\frac{1}{16}y^2 - \frac{3}{2}y + 9\right)\mathrm{d}y = \frac{1}{48}y^3 - \frac{3}{4}y^2 + 9y \ (+c)$	M1	1.1b
		A1	1.1b
	$\text{Area}(R) = \left(\frac{1}{48}(12)^3 - \frac{3}{4}(12)^2 + 9(12)\right) - (0)$ $= 36 - 108 + 108 = 36$ *	A1*	1.1b
		(8)	
	5(c) Alternative 2		
	$B\left(-4, \frac{10}{3}\right)$ into $l \Rightarrow \frac{10p}{3} = -4 + 4p^2$	M1	3.1a
	$6p^2 - 5p - 6 = 0 \Rightarrow (2p-3)(3p+2) = 0 \Rightarrow p = ...$	M1	1.1b
	$p = \frac{3}{2}$ and l cuts px-axis when $\frac{3}{2}(0) = x + 4\left(\frac{3}{2}\right)^2 \Rightarrow x = ...$	M1	2.1
	$x = -9$	A1	1.1b
	$p = \frac{3}{2} \Rightarrow P(9, 12)$ and $x = 0$ in $l : y = \frac{2}{3}x + 6$ gives $y = 6$ $\Rightarrow \text{Area}(R) = \frac{1}{2}(9)(6) + \int_0^9\left(\left(\frac{2}{3}x + 6\right) - \left(4x^{\frac{1}{2}}\right)\right)\mathrm{d}x$	M1	2.1
	$\int\left(\frac{2}{3}x + 6 - 4x^{\frac{1}{2}}\right)\mathrm{d}x = \frac{1}{3}x^2 + 6x - \frac{8}{3}x^{\frac{3}{2}} \ (+c)$	M1	1.1b
		A1	1.1b
	$\text{Area}(R) = 27 + \left(\left(\frac{1}{3}(9)^2 + 6(9) - \frac{8}{3}\left(9^{\frac{3}{2}}\right)\right) - (0)\right)$ $= 27 + (27 + 54 - 72) = 27 + 9 = 36$ *	A1*	1.1b
		(8)	
		(12 marks)	

Question 5 notes:
(a) **B1:** Substitutes $y_P = 8p$ into y^2 to obtain $64p^2$ and substitutes $x_P = 4p^2$ into $16x$ to obtain $64p^2$ and concludes that P lies on C
(b) **M1:** Uses the given formula to deduce the derivative. Alternatively, may differentiate using chain rule to deduce it **M1:** Applies $y - 8p = m(x - 4p^2)$, with their tangent gradient m, which is in terms of p. Accept use of $8p = m(4p^2) + c$ with a clear attempt to find c **A1*:** Obtains $py = x + 4p^2$ by **cso**
(c) **M1:** Substitutes their $x = "-a"$ and $y = \frac{10}{3}$ into l **M1:** Obtains a 3 term quadratic and solves (using the usual rules) to give $p = ...$ **M1:** Substitutes their p (which must be positive) and $y = 0$ into l and solves to give $x = ...$ **A1:** Finds that l cuts the x-axis at $x = -9$ **M1:** Fully correct method for finding the area of R i.e. $\frac{1}{2}(\text{their } x_P - "-9")(\text{their } y_P) - \int_0^{\text{their } x_P} 4x^{\frac{1}{2}}\,dx$ **M1:** Integrates $\pm\lambda x^{\frac{1}{2}}$ to give $\pm\mu x^{\frac{3}{2}}$, where $\lambda, \mu \neq 0$ **A1:** Integrates $4x^{\frac{1}{2}}$ to give $\frac{8}{3}x^{\frac{3}{2}}$, simplified or un-simplified **A1*:** Fully correct proof leading to a correct answer of 36
(c) Alternative 1 **M1:** Substitutes their $x = "-a"$ and $y = \frac{10}{3}$ into l **M1:** Obtains a 3 term quadratic and solves (using the usual rules) to give $p = ...$ Substitutes their p (which must be positive) into l and rearranges to give $x = ...$ **M1:** Finds l as $x = \frac{3}{2}y - 9$ **A1:** Fully correct method for finding the area of R **M1:** i.e. $\int_0^{\text{their } y_P} \left(\frac{1}{16}y^2 - \text{their}\left(\frac{3}{2}y - 9\right)\right)dy$ **M1:** Integrates $\pm\lambda y^2 \pm \mu y \pm \nu$ to give $\pm\alpha y^3 \pm \beta y^2 \pm \nu y$, where $\lambda, \mu, \nu, \alpha, \beta \neq 0$ **A1:** Integrates $\frac{1}{16}y^2 - \left(\frac{3}{2}y - 9\right)$ to give $\frac{1}{48}y^3 - \frac{3}{4}y^2 + 9y$, simplified or un-simplified **A1*:** Fully correct proof leading to a correct answer of 36

Question 5 notes continued:	
(c)	**Alternative 2**
M1:	Substitutes their $x = "-a"$ and $y = \frac{10}{3}$ into l
M1:	Obtains a 3 term quadratic and solves (using the usual rules) to give $p = ...$
M1:	Substitutes their p (which must be positive) and $y = 0$ into l and solves to give $x = ...$
A1:	Finds that l cuts the x-axis at $x = -9$
M1:	Fully correct method for finding the area of R i.e. $\frac{1}{2}(\text{their } 9)(\text{their } 6) + \int_0^{\text{their } x_P} \left(\text{their} \left(\frac{2}{3}x + 6 \right) - \left(4x^{\frac{1}{2}} \right) \right) \mathrm{d}y$
M1:	Integrates $\pm\lambda x \pm \mu \pm \nu x^{\frac{1}{2}}$ to give $\pm\alpha x^2 \pm \mu x \pm \beta x^{\frac{3}{2}}$, where $\lambda, \mu, \nu, \alpha, \beta \neq 0$
A1:	Integrates $\left(\frac{2}{3}x + 6 \right) - \left(4x^{\frac{1}{2}} \right)$ to give $\frac{1}{3}x^2 + 6x - \frac{8}{3}x^{\frac{3}{2}}$, simplified or un-simplified
A1*:	Fully correct proof leading to a correct answer of 36

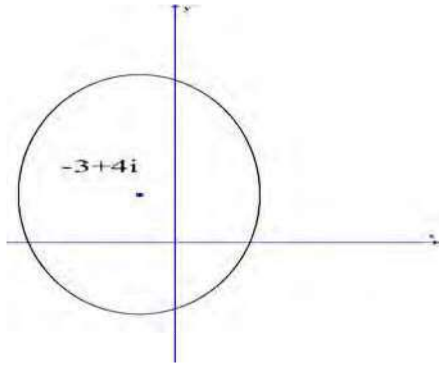
Further Pure Mathematics 2 Mark Scheme (Section B)

Question	Scheme	Marks	AOs
6(a)	Consider $\det\begin{pmatrix}3-\lambda & 1\\ 6 & 4-\lambda\end{pmatrix}=(3-\lambda)(4-\lambda)-6$	M1	1.1b
	So $\lambda^2-7\lambda+6=0$ is characteristic equation	A1	1.1b
		(2)	
	So $\mathbf{A}^2=7\mathbf{A}-6\mathbf{I}$	B1ft	1.1b
(b)	Multiplies both sides of their equation by **A** so $\mathbf{A}^3=7\mathbf{A}^2-6\mathbf{A}$	M1	3.1a
	Uses $\mathbf{A}^3=7(7\mathbf{A}-6\mathbf{I})-6\mathbf{A}$ So $\mathbf{A}^3=43\mathbf{A}-42\mathbf{I}$ *	A1*cso	1.1b
		(3)	
			(5 marks)

Notes:

(a)
M1: Complete method to find characteristic equation
A1: Obtains a correct three term quadratic equation – may use variable other than λ

(b)
B1ft: Uses Cayley Hamilton Theorem to produce equation replacing λ with **A** and constant term with constant multiple of identity matrix, **I**
M1: Multiplies equation by **A**
A1*: Replaces $\mathbf{A}^2$ by linear expression in **A** and achieves printed answer with no errors

Question	Scheme	Marks	AOs
8(a)	$(x-9)^2+(y+12)^2=4[x^2+y^2]$	M1	2.1
	$3x^2+3y^2+18x-24y-225=0$ which is the equation of a circle	A1*	2.2a
	As $x^2+y^2+6x-8y-75=0$ so $(x+3)^2+(y-4)^2=10^2$	M1	1.1b
	Giving centre at (-3, 4) and radius = 10	A1ft	1.1b
		(4)	
(b)		M1	1.1b
		A1	1.1b
		(2)	
(c)	Values range from **their** $-3-10$ to their $-3+10$	M1	3.1a
	So $-13 \le \text{Re}(w) \le 7$	A1ft	1.1b
		(2)	
			(8 marks)

Notes:

(a)

M1: Obtains an equation in terms of *x* and *y* using the given information

A1: Expands and simplifies the algebra, collecting terms and obtains a circle equation correctly, deducing that this is a circle

M1: Completes the square for their equation to find centre and radius

A1ft: Both correct

(b)

M1: Draws a circle with centre and radius as given from **their** equation

A1: Correct circle drawn, as above, with centre at $-3+4$i and passing through all four quadrants

(c)

M1: Attempts to find where a line parallel to the real axis, passing through the centre of the circle, meets the circle so using " their $-3-10$" to "their $-3+10$"

A1ft: Correctly obtains the correct answer for their centre and radius

Question	Scheme	Marks	AOs
10(a)	P_{n-1} is the population at the end of year $n-1$ and this is increased by 10% by the end of year n, so is multiplied by 110% = 1.1 to give $1.1\times P_{n-1}$ as new population by natural causes	B1	3.3
	Q is subtracted from $1.1\times P_{n-1}$ as Q is the number of deer removed from the estate	B1	3.4
	So $P_n = 1.1P_{n-1} - Q, \quad P_0 = 5000$ as population at start is 5000 and $n \in \mathbb{Z}^+$	B1	1.1b
		(3)	
(b)	Let $n = 0$, then $P_0 = (5000-10Q)(1.1)^0 + 10Q = 5000$ so result is true when $n = 0$	B1	2.1
	Assume result is true for $n = k$, $P_k = (1.1)^k(5000-10Q)+10Q$, then **as** $P_{k+1} = 1.1P_k - Q$, so $P_{k+1} = \ldots$	M1	2.4
	$P_{k+1} = 1.1\times 1.1^k(5000-10Q) + 1.1\times 10Q - Q$	A1	1.1b
	So $P_{k+1} = (5000-10Q)(1.1)^{k+1} + 10Q,$	A1	1.1b
	Implies result holds for $n = k+1$ and so by induction $P_n = (5000-10Q)(1.1)^n + 10Q,$ is true for all integer n	B1	2.2a
		(5)	
(c)	For $Q < 500$ the population of deer will grow, for $Q > 500$ the population of deer will fall	B1	3.4
	For $Q = 500$ the population of deer remains steady at 5000,	B1	3.4
		(2)	
			(10 marks)

Notes:

(a)
B1: Need to see 10% increase linked to multiplication by scale factor 1.1
B1: Needs to explain that subtraction of Q indicates the removal of Q deer from population
B1: Needs complete explanation with mention of $P_n = 1.1P_{n-1} - Q, \quad P_0 = 5000$ being the initial number of deer

(b)
B1: Begins proof by induction by considering $n = 0$
M1: Assumes result is true for $n = k$ and uses iterative formula to consider $n = k + 1$
A1: Correct algebraic statement
A1: Correct statement for $k + 1$ in required form
B1: Completes the inductive argument

(c)
B1: Consideration of both possible ranges of values for Q as listed in the scheme
B1: Gives the condition for the steady state

Answer ALL questions. Write your answers in the spaces provided.

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

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2. Prove by induction that for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

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3. $$f(z) = z^4 + az^3 + 6z^2 + bz + 65$$

where a and b are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

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8. The line l_1 has equation $\dfrac{x-2}{4} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$

The plane Π has equation $x - 2y + z = 6$

The line l_2 is the reflection of the line l_1 in the plane Π.

Find a vector equation of the line l_2

(7)

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Paper 1: Core Pure Mathematics 1 Mark Scheme

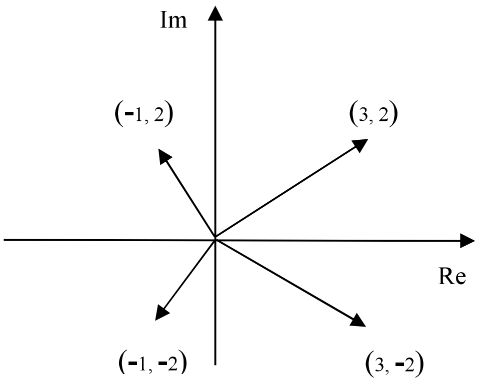
Question	Scheme	Marks	AOs
1	$\frac{1}{(r+1)(r+3)} \equiv \frac{A}{(r+1)} + \frac{B}{(r+3)} \Rightarrow A = ..., B = ...$	M1	3.1a
	$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} =$ $\frac{1}{2\times2} - \frac{1}{2\times4} + \frac{1}{2\times3} - \frac{1}{2\times5} + ... + \frac{1}{2n} - \frac{1}{2(n+2)} + \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$	M1	2.1
	$= \frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$	A1	2.2a
	$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)}$	M1	1.1b
	$= \frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	
	Alternative by induction: $n = 1 \Rightarrow \frac{1}{8} = \frac{a+b}{12\times3\times4},\ n = 2 \Rightarrow \frac{1}{8} + \frac{1}{15} = \frac{2(2a+b)}{12\times4\times5}$ $a + b = 18,\ 2a + b = 23 \Rightarrow a = ..., b = ...$	M1	3.1a
	Assume true for $n = k$ so $\sum_{r=1}^{k} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)}$		
	$\sum_{r=1}^{k+1} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)}$	M1	2.1
	$\frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)} = \frac{k(5k+13)(k+4) + 12(k+3)}{12(k+2)(k+3)(k+4)}$	A1	2.2a
	$= \frac{5k^3 + 33k^2 + 52k + 12k + 36}{12(k+2)(k+3)(k+4)} = \frac{(k+1)(k+2)(5k+18)}{12(k+2)(k+3)(k+4)}$	M1	1.1b
	$= \frac{(\underline{k+1})(5(\underline{k+1})+13)}{12(\underline{k+1}+2)(\underline{k+1}+3)}$ So true for $n = k + 1$ So $\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	
			(5 marks)

Question 1 notes:
Main Scheme **M1:** Valid attempt at partial fractions **M1:** Starts the process of differences to identify the relevant fractions at the start and end **A1:** Correct fractions that do not cancel **M1:** Attempt common denominator **A1:** Correct answer
Alternative by Induction: **M1:** Uses $n = 1$ and $n = 2$ to identify values for a and b **M1:** Starts the induction process by adding the $(k + 1)^{\text{th}}$ term to the sum of k terms **A1:** Correct single fraction **M1:** Attempt to factorise the numerator **A1:** Correct answer and conclusion

Question	Scheme	Marks	AOs
2	When $n = 1$, $2^{3n+1} + 3\left(5^{2n+1}\right) = 16 + 375 = 391$ 391 = 17 × 23 so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $2^{3k+1} + 3\left(5^{2k+1}\right)$ is divisible by 17	M1	2.4
	$\mathrm{f}(k+1) - \mathrm{f}(k) = 2^{3k+4} + 3\left(5^{2k+3}\right) - 2^{3k+1} - 3\left(5^{2k+1}\right)$	M1	2.1
	$= 7 \times 2^{3k+1} + 7 \times 3\left(5^{2k+1}\right) + 17 \times 3\left(5^{2k+1}\right)$		
	$= 7\mathrm{f}(k) + 17 \times 3\left(5^{2k+1}\right)$	A1	1.1b
	$\mathrm{f}(k+1) = 8\mathrm{f}(k) + 17 \times 3\left(5^{2k+1}\right)$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n	A1	2.4
		(6)	
			(6 marks)

Notes:

B1: Shows the statement is true for $n = 1$
M1: Assumes the statement is true for $n = k$
M1: Attempts $\mathrm{f}(k+1) - \mathrm{f}(k)$
A1: Correct expression in terms of $\mathrm{f}(k)$
A1: Correct expression in terms of $\mathrm{f}(k)$
A1: Obtains a correct expression for $\mathrm{f}(k + 1)$
A1: Correct complete conclusion

Question	Scheme	Marks	AOs
3	$z = 3 - 2\text{i}$ is also a root	B1	1.2
	$\left(z - (3 + 2\text{i})\right)\left(z - (3 - 2\text{i})\right) = \ldots$ **or** Sum of roots = 6, Product of roots = 13 $\Rightarrow \ldots$	M1	3.1a
	$= z^2 - 6z + 13$	A1	1.1b
	$\left(z^4 + az^3 + 6z^2 + bz + 65\right) = \left(z^2 - 6z + 13\right)\left(z^2 + cz + 5\right) \Rightarrow c = \ldots$	M1	3.1a
	$z^2 + 2z + 5 = 0$	A1	1.1b
	$z^2 + 2z + 5 = 0 \Rightarrow z = \ldots$	M1	1.1a
	$z = -1 \pm 2\text{i}$	A1	1.1b
		B1 $3 \pm 2\text{i}$ Plotted correctly	1.1b
		B1ft $-1 \pm 2\text{i}$ Plotted correctly	1.1b
			(9 marks)

Notes:

B1: Identifies the complex conjugate as another root
M1: Uses the conjugate pair and a correct method to find a quadratic factor
A1: Correct quadratic
M1: Uses the given quartic and their quadratic to identify the value of c
A1: Correct 3TQ
M1: Solves their second quadratic
A1: Correct second conjugate pair
B1: First conjugate pair plotted correctly and labelled
B1ft: Second conjugate pair plotted correctly and labelled (Follow through their second conjugate pair)

Question	Scheme	Marks	AOs
8	$2+4\lambda-2(4-2\lambda)-6+\lambda=6 \Rightarrow \lambda=...$	M1	1.1b
	$\lambda=2 \Rightarrow$ Required point is $(2+2(4),\ 4+2(-2),\ -6+2(1))$ $(10,\ 0,\ -4)$	A1	1.1b
	$2+t-2(4-2t)-6+t=6 \Rightarrow t=...$	M1	3.1a
	$t=3$ so reflection of $(2,4,-6)$ is $(2+6(1),4+6(-2),-6+6(1))$	M1	3.1a
	$(8,\ -8,\ 0)$	A1	1.1b
	$\begin{pmatrix}10\\0\\-4\end{pmatrix}-\begin{pmatrix}8\\-8\\0\end{pmatrix}=\begin{pmatrix}2\\8\\-4\end{pmatrix}$	M1	3.1a
	$\mathbf{r}=\begin{pmatrix}10\\0\\-4\end{pmatrix}+k\begin{pmatrix}1\\4\\-2\end{pmatrix}$ or equivalent e.g. $\left(\mathbf{r}-\begin{pmatrix}10\\0\\-4\end{pmatrix}\right)\times\begin{pmatrix}1\\4\\-2\end{pmatrix}=\mathbf{0}$	A1	2.5
		(7)	
		(7 marks)	

Notes:

M1: Substitutes the parametric equation of the line into the equation of the plane and solves for λ

A1: Obtains the correct coordinates of the intersection of the line and the plane

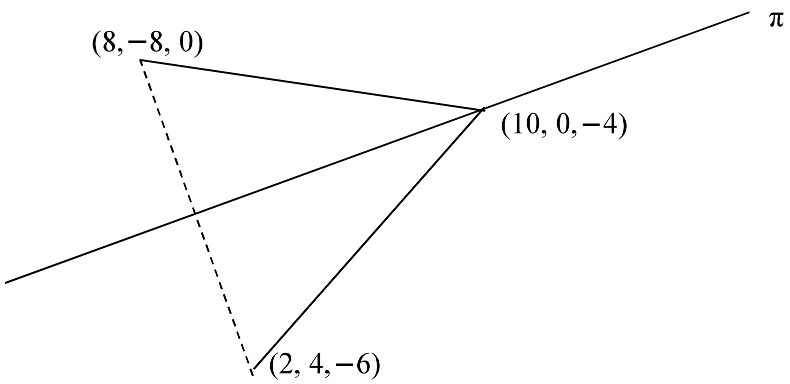
M1: Substitutes the parametric form of the line perpendicular to the plane passing through (2, 4, −6) into the equation of the plane to find t

M1: Find the reflection of (2, 4, −6) in the plane

A1: Correct coordinates

M1: Determines the direction of l by subtracting the appropriate vectors

A1: Correct vector equation using the correct notation



Answer ALL questions. Write your answers in the spaces provided.

1. The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

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2. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to Π_2

(2)

(c) Show that the acute angle between Π_1 and Π_2 is 52° to the nearest degree.

(3)

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

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4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

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5.

$$y = \sin x \sinh x$$

(a) Show that $\frac{d^4 y}{dx^4} = -4y$

(4)

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form.

(4)

(c) Find an expression for the nth non-zero term of the Maclaurin series for y.

(2)

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6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \left\{z : |z - 4 - 3\mathbf{i}| \leqslant 5\right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

Paper 2: Core Pure Mathematics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(i)	$\alpha+\beta+\gamma=8, \quad \alpha\beta+\beta\gamma+\gamma\alpha=28, \quad \alpha\beta\gamma=32$	B1	3.1a
	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$	M1	1.1b
	$=\frac{7}{8}$	A1ft	1.1b
		(3)	
(ii)	$(\alpha+2)(\beta+2)(\gamma+2)=(\alpha\beta+2\alpha+2\beta+4)(\gamma+2)$	M1	1.1b
	$=\alpha\beta\gamma+2(\alpha\beta+\alpha\gamma+\beta\gamma)+4(\alpha+\beta+\gamma)+8$	A1	1.1b
	$=32+2(28)+4(8)+8=128$	A1	1.1b
		(3)	
	Alternative:		
	$(x-2)^3-8(x-2)^2+28(x-2)-32=0$	M1	1.1b
	$=\ldots-8+\ldots-32+\ldots-56-32=-128$	A1	1.1b
	$\therefore(\alpha+2)(\beta+2)(\gamma+2)=128$	A1	1.1b
		(3)	
(iii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$	M1	3.1a
	$=8^2-2(28)=8$	A1ft	1.1b
		(2)	
		(8 marks)	

Notes:

(i)
B1: Identifies the correct values for all 3 expressions (can score anywhere)
M1: Uses a correct identity
A1ft: Correct value (follow through their 8, 28 and 32)

(ii)
M1: Attempts to expand
A1: Correct expansion
A1: Correct value

Alternative:
M1: Substitutes $x-2$ for x in the given cubic
A1: Calculates the correct constant term
A1: Changes sign and so obtains the correct value

(iii)
M1: Establishes the correct identity
A1ft: Correct value (follow through their 8, 28 and 32)

Question	Scheme	Marks	AOs
2(a)	$\begin{pmatrix}3\\-4\\2\end{pmatrix}\bullet\begin{pmatrix}6\\2\\12\end{pmatrix}=18-8+24$	M1	3.1a
	$d=\dfrac{18-8+24-5}{\sqrt{3^2+4^2+2^2}}$	M1	1.1b
	$=\sqrt{29}$	A1	1.1b
		(3)	
(b)	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\5\end{pmatrix}=\ldots$ and $\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\5\end{pmatrix}=0$ and $\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-2\end{pmatrix}=0$ $\therefore\ -\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is perpendicular to Π_2	A1	2.2a
		(2)	
(c)	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}3\\-4\\2\end{pmatrix}=-3+12+2$	M1	1.1b
	$\sqrt{(-1)^2+(-3)^2+1^2}\sqrt{(3)^2+(-4)^2+2^2}\cos\theta=11$ $\Rightarrow\cos\theta=\dfrac{11}{\sqrt{(-1)^2+(-3)^2+1^2}\sqrt{(3)^2+(-4)^2+2^2}}$	M1	2.1
	So angle between planes $\theta=52°$ *	A1*	2.4
		(3)	
		(8 marks)	

Notes:

(a)

M1: Realises the need to and so attempts the scalar product between the normal and the position vector

M1: Correct method for the perpendicular distance

A1: Correct distance

(b)

M1: Recognises the need to calculate the scalar product between the given vector and both direction vectors

A1: Obtains zero both times and makes a conclusion

(c)

M1: Calculates the scalar product between the two normal vectors

M1: Applies the scalar product formula with their 11 to find a value for cos θ

A1*: Identifies the correct angle by linking the angle between the normal and the angle between the planes

Question	Scheme		Marks	AOs
3(i)(a)	$\|\mathbf{M}\| = 2(1+2) - a(-1-1) + 4(2-1) = 0 \Rightarrow a = \ldots$		M1	2.3
	The matrix **M** has an inverse when $a \neq -5$		A1	1.1b
			(2)	
(b)	$\text{Minors}: \begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix}$ or $\text{Cofactors}: \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$		B1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{\|\mathbf{M}\|}\text{adj}(\mathbf{M})$		M1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$	2 correct rows or columns. Follow through their det**M**	A1ft	1.1b
		All correct. Follow through their det**M**	A1ft	1.1b
			(4)	
(ii)	When $n = 1$, lhs $= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$, rhs $= \begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ So the statement is true for $n = 1$		B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$		M1	2.4
	$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$		M1	2.1
	$= \begin{pmatrix} 3\times 3^k & 0 \\ 3\times 3(3^k-1)+6 & 1 \end{pmatrix}$		A1	1.1b
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$		A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n		A1	2.4
			(6)	
			(12 marks)	

Question 3 notes:	
(i)(a)	
M1:	Attempts determinant, equates to zero and attempts to solve for a in order to establish the restriction for a
A1:	Provides the correct condition for a if **M** has an inverse
(i)(b)	
B1:	A correct matrix of minors or cofactors
M1:	For a complete method for the inverse
A1ft:	Two correct rows following through their determinant
A1ft:	Fully correct inverse following through their determinant
(ii)	
B1:	Shows the statement is true for $n = 1$
M1:	Assumes the statement is true for $n = k$
M1:	Attempts to multiply the correct matrices
A1:	Correct matrix in terms of k
A1:	Correct matrix in terms of $k + 1$
A1:	Correct complete conclusion

Question	Scheme	Marks	AOs
4(a)	$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1	2.1
	$= 2\cos n\theta$ *	A1*	1.1b
		(2)	
(b)	$\left(z + z^{-1}\right)^4 = 16\cos^4\theta$	B1	2.1
	$\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1	2.1
	$= z^4 + z^{-4} + 4\left(z^2 + z^{-2}\right) + 6$	A1	1.1b
	$= 2\cos 4\theta + 4\left(2\cos 2\theta\right) + 6$	M1	2.1
	$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$*	A1*	1.1b
		(5)	
			(7 marks)

Notes:

(a)
M1: Identifies the correct form for z^n and z^{-n} and adds to progress to the printed answer
A1*: Achieves printed answer with no errors

(b)
B1: Begins the argument by using the correct index with the result from part (a)
M1: Realises the need to find the expansion of $\left(z + z^{-1}\right)^4$
A1: Terms correctly combined
M1: Links the expansion with the result in part (a)
A1*: Achieves printed answer with no errors

Question	Scheme	Marks	AOs
5(a)	$\dfrac{dy}{dx} = \sin x\cosh x + \cos x\sinh x$	M1	1.1a
	$\dfrac{d^2y}{dx^2} = \cos x\cosh x + \sin x\sinh x + \cos x\cosh x - \sin x\sinh x$ $(= 2\cos x\cosh x)$	M1	1.1b
	$\dfrac{d^3y}{dx^3} = 2\cos x\sinh x - 2\sin x\cosh x$	M1	1.1b
	$\dfrac{d^4y}{dx^4} = -4\sinh x\sin x = -4y$ *	A1*	2.1
		(4)	
(b)	$\left(\dfrac{d^2y}{dx^2}\right)_0 = 2,\ \left(\dfrac{d^6y}{dx^6}\right)_0 = -8,\ \left(\dfrac{d^{10}y}{dx^{10}}\right)_0 = 32$	B1	3.1a
	Uses $y = y_0 + xy_0' + \dfrac{x^2}{2!}y_0'' + \dfrac{x^3}{3!}y_0''' + \ldots$ with their values	M1	1.1b
	$= \dfrac{x^2}{2!}(2) + \dfrac{x^6}{6!}(-8) + \dfrac{x^{10}}{10!}(32)$	A1	1.1b
	$= x^2 - \dfrac{x^6}{90} + \dfrac{x^{10}}{113400}$	A1	1.1b
		(4)	
(c)	$2(-4)^{n-1}\dfrac{x^{4n-2}}{(4n-2)!}$	M1 A1	3.1a 2.2a
		(2)	
		(10 marks)	

Notes:

(a)

M1: Realises the need to use the product rule and attempts first derivative

M1: Realises the need to use a second application of the product rule and attempts the second derivative

M1: Correct method for the third derivative

A1*: Obtains the correct 4th derivative and links this back to y

(b)

B1: Makes the connection with part **(a)** to establish the general pattern of derivatives and finds the correct non-zero values

M1: Correct attempt at Maclaurin series with their values

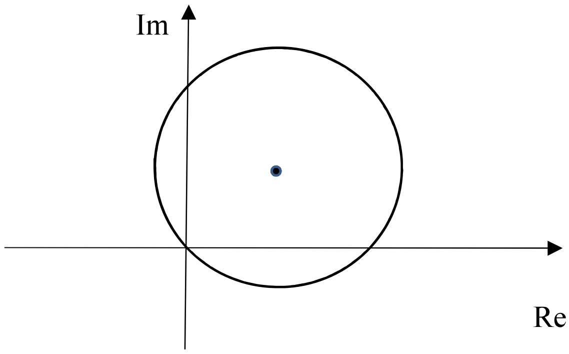
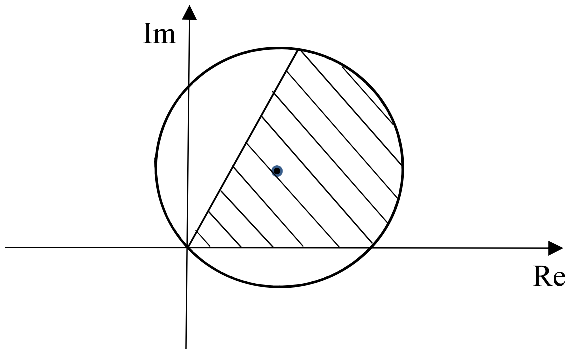
A1: Correct expression un-simplified

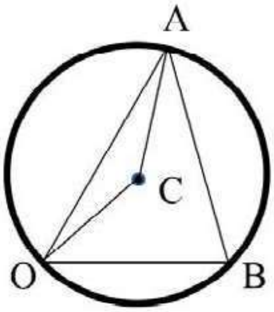
A1: Correct expression and simplified

(c)

M1: Generalising, dealing with signs, powers and factorials

A1: Correct expression

Question	Scheme	Marks	AOs
6(a)(i)		M1	1.1b
		A1	1.1b
(a)(ii)	$\lvert z-4-3\mathrm{i}\rvert=5\Rightarrow\lvert x+\mathrm{i}y-4-3\mathrm{i}\rvert=5\Rightarrow(x-4)^2+(y-3)^2=...$	M1	2.1
	$(x-4)^2+(y-3)^2=25$ or any correct form	A1	1.1b
	$(r\cos\theta-4)^2+(r\sin\theta-3)^2=25$ $\Rightarrow r^2\cos^2\theta-8r\cos\theta+16+r^2\sin^2\theta-6r\sin\theta+9=25$ $\Rightarrow r^2-8r\cos\theta-6r\sin\theta=0$	M1	2.1
	$\therefore r=8\cos\theta+6\sin\theta$ *	A1*	2.2a
		(6)	
(b)(i)		B1	1.1b
		B1ft	1.1b
(b)(ii)	$A=\frac{1}{2}\int r^2\,\mathrm{d}\theta=\frac{1}{2}\int(8\cos\theta+6\sin\theta)^2\,\mathrm{d}\theta$ $=\frac{1}{2}\int\left(64\cos^2\theta+96\sin\theta\cos\theta+36\sin^2\theta\right)\mathrm{d}\theta$	M1	3.1a
	$=\frac{1}{2}\int\left(32(\cos2\theta+1)+96\sin\theta\cos\theta+18(1-\cos2\theta)\right)\mathrm{d}\theta$	M1	1.1b
	$=\frac{1}{2}\int(14\cos2\theta+50+48\sin2\theta)\,\mathrm{d}\theta$	A1	1.1b
	$=\frac{1}{2}\left[7\sin2\theta+50\theta-24\cos2\theta\right]_0^{\frac{\pi}{3}}=\frac{1}{2}\left\{\left(\frac{7\sqrt{3}}{2}+\frac{50\pi}{3}+12\right)-(-24)\right\}$	M1	2.1
	$=\frac{7\sqrt{3}}{4}+\frac{25\pi}{3}+18$	A1	1.1b
		(7)	

Question	Scheme	Marks	AOs
	(b)(ii) Alternative:  Candidates may take a geometric approach e.g. by finding sector + 2 triangles		
	Angle $ACB = \left(\frac{2\pi}{3}\right)$ so area sector $ACB = \frac{1}{2}(5)^2\frac{2\pi}{3}$ Area of triangle $OCB = \frac{1}{2}\times 8\times 3$	M1	3.1a
	Sector area ACB + triangle area $OCB = \frac{25\pi}{3} + 12$	A1	1.1b
	Area of triangle OAC: Angle $ACO = 2\pi - \frac{2\pi}{3} - \cos^{-1}\left(\frac{5^2+5^2-8^2}{2\times5\times5}\right)$ so area $OAC = \frac{1}{2}(5)^2\sin\left(\frac{4\pi}{3} - \cos^{-1}\left(\frac{-7}{25}\right)\right)$	M1	1.1b
	$= \frac{25}{2}\left(\sin\frac{4\pi}{3}\cos\left(\cos^{-1}\left(\frac{-7}{25}\right)\right) - \cos\frac{4\pi}{3}\sin\left(\cos^{-1}\left(\frac{-7}{25}\right)\right)\right)$ $= \frac{25}{2}\left(\left(\frac{7\sqrt{3}}{50}\right) + \frac{1}{2}\sqrt{1-\left(\frac{7}{25}\right)^2}\right) = \frac{7\sqrt{3}}{4} + 6$ Total area $= \frac{25\pi}{3} + \frac{1}{2}\times 8\times 3 + 6 + \frac{7\sqrt{3}}{4}$	M1	2.1
	$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$	A1	1.1b
			(13 marks)

Question 6 notes:	
(a)(i)	
M1:	Draws a circle which passes through the origin
A1:	Fully correct diagram
(a)(ii)	
M1:	Uses $z = x + \mathrm{i}y$ in the given equation and uses modulus to find equation in x and y only
A1:	Correct equation in terms of x and y in any form – may be in terms of r and θ
M1:	Introduces polar form, expands and uses $\cos^2\theta + \sin^2\theta = 1$ leading to a polar equation
A1*:	Deduces the given equation (ignore any reference to $r = 0$ which gives a point on the curve)
(b)(i)	
B1:	Correct pair of rays added to their diagram
B1ft:	Area between their pair of rays and inside their circle from (a) shaded, as long as there is an intersection
(b)(ii)	
M1:	Selects an appropriate method by linking the diagram to the polar curve in (a), evidenced by use of the polar area formula
M1:	Uses double angle identities
A1:	Correct integral
M1:	Integrates and applies limits
A1:	Correct area
(b)(ii) Alternative:	
M1:	Selects an appropriate method by finding angle ACB and area of sector ACB and finds area of triangle OCB to make progress towards finding the required area
A1:	Correct combined area of sector ACB + triangle OCB
M1:	Starts the process of finding the area of triangle OAC by calculating angle ACO and attempts area of triangle OAC
M1:	Uses the addition formula to find the exact area of triangle OAC and employs a full correct method to find the area of the shaded region
A1:	Correct area

Answer **all** questions in the spaces provided.

1 A reflection is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

State the equation of the line of invariant points.

Circle your answer.

[1 mark]

$x = 0$ $\quad$ $y = 0$ $\quad$ $y = x$ $\quad$ $y = -x$

3 Find the equations of the asymptotes of the curve $x^2 - 3y^2 = 1$

Circle your answer.

[1 mark]

$y = \pm\, 3x$ $\quad\quad$ $y = \pm\frac{1}{3}x$ $\quad\quad$ $y = \pm\sqrt{3}\,x$ $\quad\quad$ $y = \pm\frac{1}{\sqrt{3}}x$

Turn over for the next question

4 $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4 (a) Find the value of k for which matrix **A** is singular.

[1 mark]

4 (b) Describe the transformation represented by matrix **B**.

[1 mark]

4 (c) (i) Given that **A** and **B** are both non-singular, verify that $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$.

[4 marks]

4 (c) (ii) Prove the result $\mathbf{M}^{-1}\mathbf{N}^{-1} = (\mathbf{NM})^{-1}$ for all non-singular square matrices **M** and **N** of the same size.

[4 marks]

Turn over for the next question

6 (a) Use the definitions of $\sinh x$ and $\cosh x$ in terms of e^{x} and e^{-x} to show that

$x = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ where $t = \tanh x$

[4 marks]

Question 6 continues on the next page

6 (b) (i) Prove $\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$

[4 marks]

6 (b) (ii) Show that the equation $\cosh 3x = 13\cosh x$ has only one positive solution.

Find this solution in exact logarithmic form.

[4 marks]

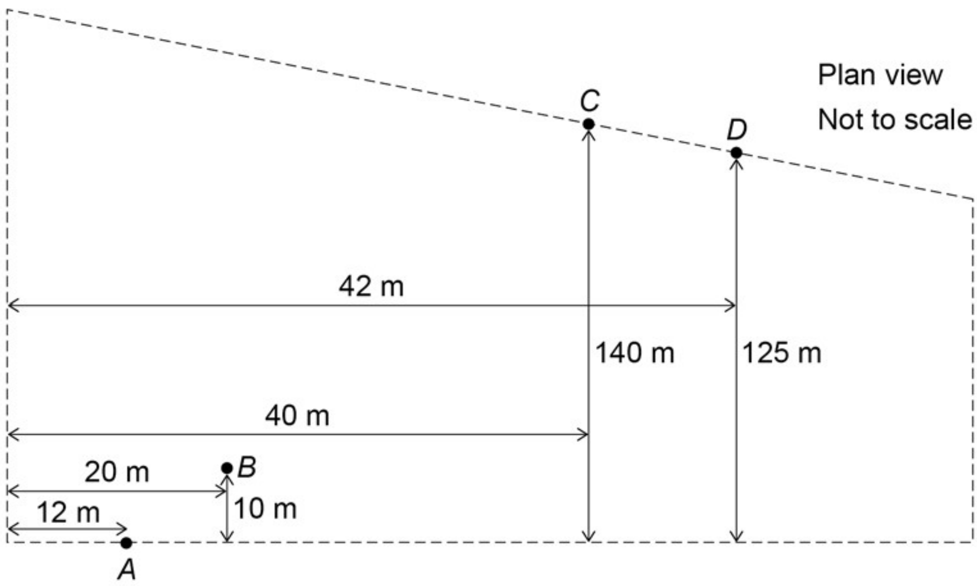
Turn over for the next question

7 A lighting engineer is setting up part of a display inside a large building. The diagram shows a plan view of the area in which he is working.

He has two lights, which project narrow beams of light.

One is set up at a point 3 metres above the point *A* and the beam from this light hits the wall 23 metres above the point *D*.

The other is set up 1 metre above the point *B* and the beam from this light hits the wall 29 metres above the point *C*.



7 (a) By creating a suitable model, show that the beams of light intersect.

[6 marks]

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7 (b) Find the angle between the two beams of light.

[3 marks]

7 (c) State one way in which the model you created in part **(a)** could be refined.

[1 mark]

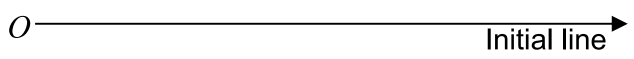
8 A curve has polar equation $r = 3 + 2\cos\theta$, where $0 \le \theta < 2\pi$

8 (a) (i) State the maximum and minimum values of r.

[2 marks]

8 (a) (ii) Sketch the curve.

[2 marks]



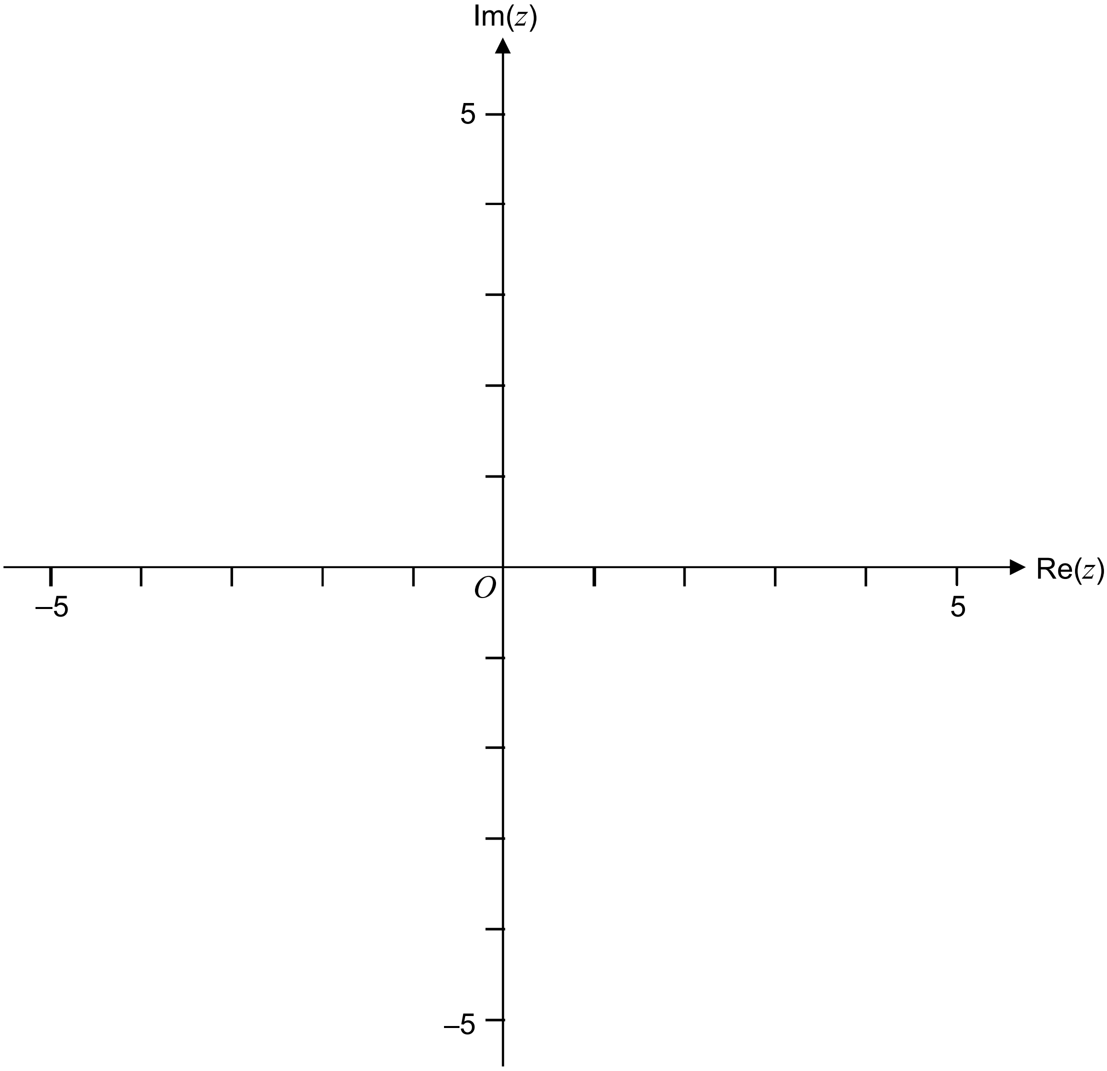
8 (b) The curve $r = 3 + 2\cos\theta$ intersects the curve with polar equation $r = 8\cos^2\theta$, where $0 \leq \theta < 2\pi$

Find all of the points of intersection of the two curves in the form $[r, \theta]$.

[6 marks]

9 (a) Sketch on the Argand diagram below, the locus of points satisfying the equation $|z - 2| = 2$

[2 marks]



9 (b) Given that $|z - 2| = 2$ and $\arg(z - 2) = -\dfrac{\pi}{3}$, express z in the form $a + b\mathrm{i}$, where a and b are real numbers.

[3 marks]

Turn over for the next question

10 (a) Prove that

$$6 + 3\sum_{r=1}^{n}(r + 1)(r + 2) = (n + 1)(n + 2)(n + 3)$$

[6 marks]

10 (b) Alex substituted a few values of n into the expression $(n + 1)(n + 2)(n + 3)$ and made the statement:

"For all positive integers n,

$$6 + 3\sum_{r=1}^{n}(r + 1)(r + 2)$$

is divisible by 12."

Disprove Alex's statement.

[2 marks]

Turn over for the next question

11 The equation $x^3 - 8x^2 + cx + d = 0$ where c and d are real numbers, has roots α, β, γ.

When plotted on an Argand diagram, the triangle with vertices at α, β, γ has an area of 8.

Given $\alpha = 2$, find the values of c and d.

Fully justify your solution.

[5 marks]

12 A curve, C_1 has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{5x^2 - 12x + 12}{x^2 + 4x - 4}$

The line $y = k$ intersects the curve, C_1

12 (a) (i) Show that $(k + 3)(k - 1) \geq 0$

[5 marks]

Question 12 continues on the next page

12 (a) (ii) Hence find the coordinates of the stationary point of C_1 that is a maximum point.

[4 marks]

12 (b) Show that the curve C_2 whose equation is $y = \dfrac{1}{f(x)}$, has no vertical asymptotes.

[2 marks]

12 (c) State the equation of the line that is a tangent to both C_1 and C_2 .

[1 mark]

END OF QUESTIONS

Q	Marking Instructions	AO	Mark	Typical Solution
1	Circles correct answer	AO1.1b	B1	$y = 0$
	Total		**1**	
3	Circles correct answer	AO1.1b	B1	$y = \pm\frac{1}{\sqrt{3}}x$
	Total		**1**	
4(a)	Finds $k = 2$	AO1.1b	B1	$k - 2 = 0 \Rightarrow k = 2$
4(b)	States correct transformation	AO1.2	B1	Reflection in the y-axis
4(c)(i)	Finds product **BA** Allow one slip	AO1.1a	M1	$\mathbf{BA} = \begin{bmatrix} -1 & -2 \\ 1 & k \end{bmatrix}$
	Obtains inverse FT 'their' **BA** provided M1 awarded	AO1.1b	A1F	$(\mathbf{BA})^{-1} = \frac{1}{-k-(-2)} \begin{bmatrix} k & 2 \\ -1 & -1 \end{bmatrix}$
	Finds $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$	AO1.1b	B1	$\mathbf{A}^{-1} = \frac{1}{k-2} \begin{bmatrix} k & -2 \\ -1 & 1 \end{bmatrix}$
	Obtains correct $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and shows that $(k-2)\times(-1) = 2-k = -k-(-2)$ thus completing verification Must clearly show $\mathbf{A}^{-1} \times \mathbf{B}^{-1}$ method for this mark – disallow if answer simply stated	AO2.1	R1	$\mathbf{B}^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\mathbf{A}^{-1}\mathbf{B}^{-1} = \frac{1}{(k-2)\times(-1)}$ $\begin{bmatrix} k+0 & (-2\times -1)+0 \\ (-1\times 1)+0 & 0+(-1\times 1) \end{bmatrix}$ That is the same as $(\mathbf{BA})^{-1}$ $(\mathbf{BA})^{-1} = \frac{1}{2-k} \begin{bmatrix} k & 2 \\ -1 & -1 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

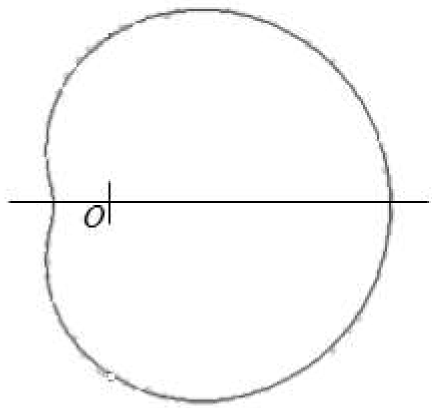
Q	Marking Instructions	AO	Mark	Typical Solution
4(c)(ii)	Uses equation for identity from definition	AO3.1a	M1	We require to demonstrate that: $(\mathbf{NM}) \times \{\mathbf{M}^{-1}\mathbf{N}^{-1}\} = \boldsymbol{I}$
	Comences argument by manipulating the matrix products within the equation with clear pairing	AO2.1	R1	$(\mathbf{NM}) \times \mathbf{M}^{-1}\mathbf{N}^{-1} = \mathbf{N}(\mathbf{M} \times \mathbf{M}^{-1})\mathbf{N}^{-1}$ $= \mathbf{N}\,\boldsymbol{I}\,\mathbf{N}^{-1}$ $= \mathbf{NN}^{-1}$
	Clearly demonstrates that $\mathbf{M} \times \mathbf{M}^{-1} = \boldsymbol{I}$ used	AO2.4	B1	$= \boldsymbol{I}$
	Completes the argument using rigorous reasoning with definition of matrix inverse and associativity mentioned Must see all working with correct pairing of each matrix with inverse	AO2.1	R1	Using definition of matrix inverse and associativity of matrix multiplication Hence true for all non-singular matrices **N** and **M**
	Total		**10**	

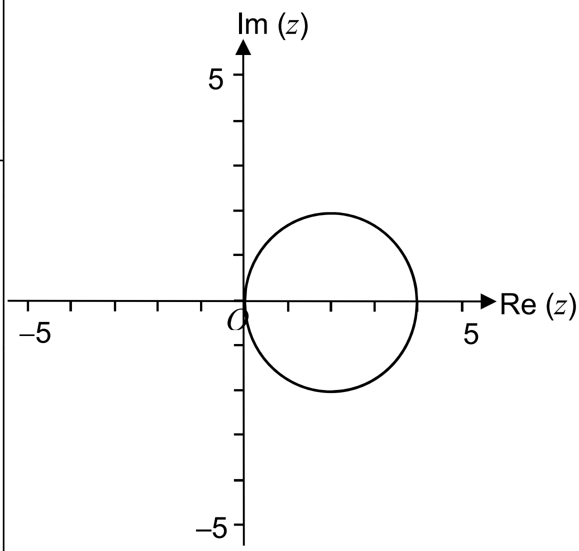
Q	Marking Instructions	AO	Mark	Typical Solution
6(a)	Uses definitions of sinh x and cosh x to obtain expression for tanh x	AO1.2	B1	$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$
	Multiplies by e^{x}	AO1.1a	M1	$\cosh x = \dfrac{e^{x} - e^{-x}}{2}$
	Obtains e^{2x}	AO1.1b	A1	$\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$
	Completes a fully correct argument by demonstrating result by taking logs This mark is available only if all previous marks have been awarded	AO2.1	R1	Multiplying numerator and denominator by e^{x} $t = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ $te^{2x} + t = e^{2x} - 1$ [or multiplies by e^{x} in $te^{x} + te^{-x} = e^{x} - e^{-x}$] $1 + t = e^{2x}(1 - t)$ $e^{2x} = \dfrac{1+t}{1-t}$ $2x = \ln\dfrac{1+t}{1-t}$ hence $x = \dfrac{1}{2}\ln\dfrac{1+t}{1-t}$

Q	Marking Instructions	AO	Mark	Typical Solution
6(b)(i)	Expresses cosh $3x$ and cosh x in exponential form Seen anywhere in solution	AO1.2	B1	To be proven: $\left(\frac{e^x+e^{-x}}{2}\right)^3 =$ $\frac{1}{4}\left(\frac{e^{3x}+e^{-3x}}{2}\right)+\frac{3}{4}\left(\frac{e^x+e^{-x}}{2}\right)$
	Expands LHS FT 'their' LHS provided first M1 awarded Allow one slip	AO1.1a	M1	LHS $\left(\frac{e^x+e^{-x}}{2}\right)^3 =$
	Simplifies and collects terms FT 'their' expression Allow one slip	AO1.1a	M1	$\frac{1}{8}(e^{3x}+3e^{2x}.e^{-x}+3e^{x}.e^{-2x}+e^{-3x}) =$
	Completes fully correct proof to reach the required result This mark is available only if all previous marks have been awarded	AO2.1	R1	$\frac{1}{4}\left(\frac{(e^{3x}+e^{-3x})}{2}\right)+3\frac{(e^x+e^{-x})}{2} =$ $\frac{1}{4}\cosh 3x+\frac{3}{4}\cosh x = \text{RHS}$ From the definition of cosh x
6(b)(ii)	Substitutes for cosh $3x$ in equation from part **(b)(i)** Allow one slip	A03.1a	M1	$(\cosh x)^3 = \frac{1}{4}\times 13\cosh x+\frac{3}{4}\cosh x$ $(\cosh x)^3 - 4\cosh x = 0$
	Obtains equation in cosh x and solves it Allow one slip	AO1.1a	M1	$\cosh x\left[(\cosh x)^2 - 4\right] = 0$ Solutions are $\cosh x = 0, -2, 2$
	Eliminates 0 and –2 with reason	AO2.4	E1	solutions 0 and -2 are not possible since range of $\cosh x \geq 1$
	States correct solution in exact log form	AO1.1b	A1	$\cosh x = 2 \Rightarrow x = \ln\left(2+\sqrt{3}\right)$
	Total		**12**	

Q	Marking Instructions	AO	Mark	Typical Solution
7(a)	Models light beams as straight lines and forms vector equations for straight lines using a suitable origin	AO3.3	M1	Modelling beams of light as straight lines taking the origin as point *A*: $\mathbf{r}_A = \begin{pmatrix}0\\0\\3\end{pmatrix} + \lambda\left(\begin{pmatrix}30\\125\\23\end{pmatrix} - \begin{pmatrix}0\\0\\3\end{pmatrix}\right)$
	Forms correct vector equation for a line. Allow one slip	AO1.1b	A1	$= \begin{pmatrix}0\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}30\\125\\20\end{pmatrix}$
	Forms correct vector equation for second line. Allow one slip	AO1.1b	A1	$\mathbf{r}_B = \begin{pmatrix}8\\10\\1\end{pmatrix} + \mu\left(\begin{pmatrix}28\\140\\29\end{pmatrix} - \begin{pmatrix}8\\10\\1\end{pmatrix}\right)$
	Forms equations for two components using 'their' model FT 'their' lines	AO3.4	M1	$= \begin{pmatrix}8\\10\\1\end{pmatrix} + \mu\begin{pmatrix}20\\130\\28\end{pmatrix}$ $30\lambda = 8 + 20\mu$
	Solves 'their' equations correctly FT 'their' lines	AO1.1b	A1F	$125\lambda = 10 + 130\mu$ $\lambda = \frac{3}{5}$ and $\mu = \frac{1}{2}$
	Checks with third component and concludes that the beams of light intersect This mark is available only if all previous marks have been awarded	AO2.1	R1	$3 + \frac{3}{5} \times 20 = 15$ $1 + \frac{1}{2} \times 28 = 15$ $\therefore$ Intersect

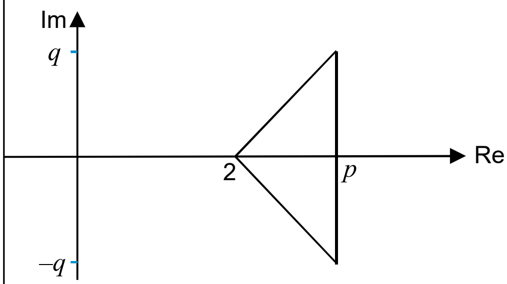
Q	Marking Instructions	AO	Mark	Typical Solution
7(b)	Evaluates scalar product for 'their' direction vectors. (PI)	AO3.1a	M1	$\begin{pmatrix}30\\125\\20\end{pmatrix}\bullet\begin{pmatrix}20\\130\\28\end{pmatrix}=17410$
	Sets up equation to find angle. (PI) FT only if previous M1 awarded	AO1.1a	M1	$\cos\theta=\dfrac{17410}{\sqrt{30^2+125^2+20^2}\times\sqrt{20^2+130^2+28^2}}$ $\cos\theta=\dfrac{17410}{\sqrt{16925}\times\sqrt{18084}}=0.9951$ $\theta=5.6°$
	Obtains correct angle.	AO1.1b	A1	
7(c)	States appropriate refinement.	AO3.5c	E1	Take account of the width of the beams.
	Total		**10**	

Q	Marking Instructions	AO	Mark	Typical Solution
8(a)(i)	States max value for r	AO1.1b	B1	Maximum value of $r = 5$
	States min value for r	AO1.1b	B1	Minimum value of $r = 1$
(ii)	Draws simple closed curve enclosing pole	AO1.1a	M1	
	Draws correct shape with dimple (not cusp) when $\theta = \pi$	AO1.1b	A1	
(b)	Equates $3 + 2\cos\theta = 8\cos^2\theta$	AO1.1a	M1	$3 + 2\cos\theta = 8\cos^2\theta$
	Solves 'their' quadratic equation FT 'their' equation only if M1 has been awarded	AO1.1a	M1	$8\cos^2\theta - 2\cos\theta - 3 = 0$ $(4\cos\theta - 3)(2\cos\theta + 1) = 0$
	Obtains 2 values for θ for each value of $\cos\theta$ FT 'their' equation only if both M1 marks have been awarded	AO1.1b	A1F	$\cos\theta = \frac{3}{4}$, $\cos\theta = -\frac{1}{2}$
	Substitutes 'their' $\cos\theta$ into a polar equation to find a value of r FT 'their' $\cos\theta$ only if both M1 marks have been awarded	AO1.1a	M1	$\theta = 0.723$ or $\frac{2\pi}{3}$ $\theta = 5.56$ or $\frac{4\pi}{3}$
	Obtains both values of r correct for 'their' $\cos\theta$ values FT 'their' $\cos\theta$ only if both M1 marks have been awarded	AO1.1b	A1F	$\cos\theta = \frac{3}{4} \Rightarrow r = \frac{9}{2}$ $\cos\theta = -\frac{1}{2} \Rightarrow r = 2$
	Deduces that four values for θ exist and expresses points in required form	AO2.2a	R1	Intersection points $\left[\frac{9}{2}, 0.723\right]$, $\left[\frac{9}{2}, 5.56\right]$, $\left[2, \frac{2\pi}{3}\right]$, $\left[2, \frac{4\pi}{3}\right]$
	Total		**10**	

Q	Marking Instructions	AO	Mark	Typical Solution
9(a)	Draws 'circle' with centre 2 + 0i Ignore other features	AO1.1a	M1	
	Draws circle passing through (0, 0), (4, 0), close to (2, 2) and $(2, -2)$ with Imaginary axis tangential	AO1.1b	A1	
(b)	Uses mod/arg forms	AO3.1a	M1	$z-2=2\left[\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]$
	Substitutes exact values for cos and sin Allow one slip	AO1.1a	M1	$=2\left[\frac{1}{2}+\mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right]$
	Obtains result in exact form	AO1.1b	A1	$z=3-\sqrt{3}\,\mathrm{i}$
	Total		**5**	

Q	Marking Instructions	AO	Mark	Typical Solution
10(a)	Splits up the sum into separate sums $\sum ar^2 + \sum br + (\sum c)$ PI	AO3.1a	M1	$\sum_{r=1}^{n}(r+1)(r+2) = \sum_{r=1}^{n}(r^2+3r+2)$ $= \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 3r + \sum_{r=1}^{n} 2$
	Substitutes for the two sums $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ Allow one slip	AO1.1a	M1	S $= \frac{n}{6}(n+1)(2n+1) + 3\frac{n}{2}(n+1) + \sum_{r=1}^{n} 2$
	States or uses $\sum_{r=1}^{n} 1 = n$ PI	AO1.2	B1	$= \frac{n}{6}(n+1)(2n+1) + 3\frac{n}{2}(n+1) + 2n$ Now $6 + 3\sum_{r=1}^{n}(r+1)(r+2)$
	Factorises out $(n + 1)$ Allow one slip	AO1.1a	M1	$= 6 + \frac{n}{2}(n+1)(2n+1) + 9\frac{n}{2}(n+1) + 6n$
	Simplifies $(n+1)\{\frac{n}{2}(2n+1) + \frac{9n}{2} + 6\}$ to find second linear factor from ‘their’ quadratic FT ‘their’ quadratic provided all M1 marks have been awarded Allow one slip	AO1.1a	M1	$= \frac{n}{2}(n+1)(2n+1) + \frac{9n}{2}(n+1) + 6(n+1)$ $= (n+1)\{\frac{n}{2}(2n+1) + \frac{9n}{2} + 6\}$ $= (n+1)(n^2+5n+6)$
	Completes a rigorous argument to show the required result To obtain this mark factorising must be clearly seen and all previous marks obtained	AO2.1	R1	$= (n+1)(n+2)(n+3)$

Q	Marking Instructions	AO	Mark	Typical Solution
10(b)	Chooses a multiple of 4 for n and obtains a correct numerical value/expression	AO2.4	E1	When $n = 4$, $6+3\sum_{r=1}^{n}(r+1)(r+2) = (5)(6)(7)$
	Clear argument with concluding statement	AO2.3	E1	= 210 which is not a multiple of 12 so Alex's statement is false.
	Total		**8**	

Q	Marking Instructions	AO	Mark	Typical Solution
11	Writes β and γ in the form $p \pm q\mathrm{i}$ (seen anywhere in the solution)	AO2.5	B1	Real coefficients $\Rightarrow \beta = p + q\mathrm{i}$ and $\gamma = p - q\mathrm{i}$
	Uses "sum of the roots = –b/a" together with a conjugate pair to determine the real part (p) of β and γ	AO3.1a	M1	$\alpha + \beta + \gamma = 8$ $\Rightarrow 2 + p + q\mathrm{i} + p - q\mathrm{i} = 8$ $\Rightarrow 2 + 2p = 8$ $\Rightarrow p = 3$
	Uses '(their p)' –2 and the area of the triangle on an Argand diagram to determine the imaginary parts of β and γ	AO3.1a	M1	$(p-2)q = 8$ $\Rightarrow q = 8$ 
	Uses a correct method to find the value of c or d using 'their' values of $p \pm q\mathrm{i}$	AO1.1a	M1	$\beta = 3 + 8\mathrm{i}$ and $\gamma = 3 - 8\mathrm{i}$
	Obtains correct values for c and d. CAO	AO1.1b	A1	$d = -\alpha\beta\gamma = -146$ $c = \sum \alpha\beta = 85$
	Total		**5**	

Q	Marking Instructions	AO	Mark	Typical Solution
12(a)(i)	Eliminates y	AO1.1a	M1	$k = \dfrac{5x^2 - 12x + 12}{x^2 + 4x - 4}$
	Obtains a quadratic equation in the form $Ax^2 + Bx + C = 0$, PI by later work	AO3.1a	M1	$k\left(x^2 + 4x - 4\right) = 5x^2 - 12x + 12$ $(k-5)x^2 + 4(k+3)x - 4(k+3) = 0$ (A)
	Obtains $b^2 - 4ac$ in terms of k for 'their' quadratic FT 'their' quadratic provided first M1 awarded	AO1.1b	A1F	$y = k$ intersects C_1 so roots of (A) are real $b^2 - 4ac =$
	Obtains inequality, including ≥ 0, where k is the only unknown for 'their' discriminant FT 'their' discriminant provided both M1 marks have been awarded	AO1.1a	M1	$[4(k+3)]^2 - 4(k-5)(-4(k+3))$ $16\left(k+3\right)^2 + 16\left(k-5\right)\left(k+3\right) \geq 0$ $16\left(k+3\right)\left(k+3+k-5\right) \geq 0$
	Completes a rigorous argument to show that $\left(k+3\right)\left(k-1\right) \geq 0$ This mark is available only if all previous marks have been awarded	AO2.1	R1	$\Rightarrow \left(k+3\right)\left(2k-2\right) \geq 0$ $\Rightarrow \left(k+3\right)\left(k-1\right) \geq 0$
12(a)(ii)	Obtains critical values	AO1.1b	B1	Critical values are -3 and 1
	Deduces that $k = -3$ for maximum	AO2.2a	R1	$k \leq -3$ (or $k \geq 1$) satisfy inequality For max pt, $k = -3$
	Substitutes for k into 'their' quadratic from **(a)(i)** FT 'their' quadratic only if first M1 awarded in **(a)(i)**	AO1.1a	M1	Sub $k = -3$ in (A) gives $-8x^2 = 0$
	States coordinates of max pt NMS 0/4 Must be using **(a)(i)**	AO1.1b	A1 CAO	Max pt of C_1 is $(0, -3)$

Q	Marking Instructions	AO	Mark	Typical Solution
12(b)	Uses discriminant to determine solution	AO2.4	E1	$(-12)^2 - 4(5)(12) < 0$
	Deduces no vertical asymptotes with clear reasoning with reference to denominator	AO2.2a	R1	$k \neq 0$ Denominator, $5x^2 - 12x + 12$ of $\frac{1}{f(x)}$ is never 0 so C_2 has no vertical asymptotes.
12 (c)	Obtains $y = 1$	AO3.2a	B1	$y = 1$
	Total		**12**	
	TOTAL		**80**	

Answer **all** questions in the spaces provided.

1 A vector is given by **a** = $\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$

Which vector is **not** perpendicular to **a**?

Circle your answer.

[1 mark]

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

2 Use the definitions of cosh x and sinh x in terms of e^x and e^{-x} to show that $\cosh^2 x - \sinh^2 x \equiv 1$

[2 marks]

3 (a) Given that

$$\frac{2}{(r+1)(r+2)(r+3)} \equiv \frac{A}{(r+1)(r+2)} + \frac{B}{(r+2)(r+3)}$$

find the values of the integers A and B

[2 marks]

3 (b) Use the method of differences to show clearly that

$$\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)} = \frac{89}{19800}$$

[4 marks]

5 $p(z) = z^4 + 3z^2 + az + b$, $a \in \mathbb{R}, b \in \mathbb{R}$

$2 - 3i$ is a root of the equation $p(z) = 0$

5 (a) Express $p(z)$ as a product of quadratic factors with real coefficients.

[5 marks]

5 (b) Solve the equation $p(z) = 0$.

[1 mark]

7 Three planes have equations,

$$x - y + kz = 3$$

$$kx - 3y + 5z = -1$$

$$x - 2y + 3z = -4$$

Where k is a real constant. The planes do not meet at a unique point.

7 (a) Find the possible values of k

[3 marks]

7 (b) There are two possible geometric configurations of the given planes.

Identify each possible configurations, stating the corresponding value of k

Fully justify your answer.

[5 marks]

7 (c) Given further that the equations of the planes form a consistent system, find the solution of the system of equations.

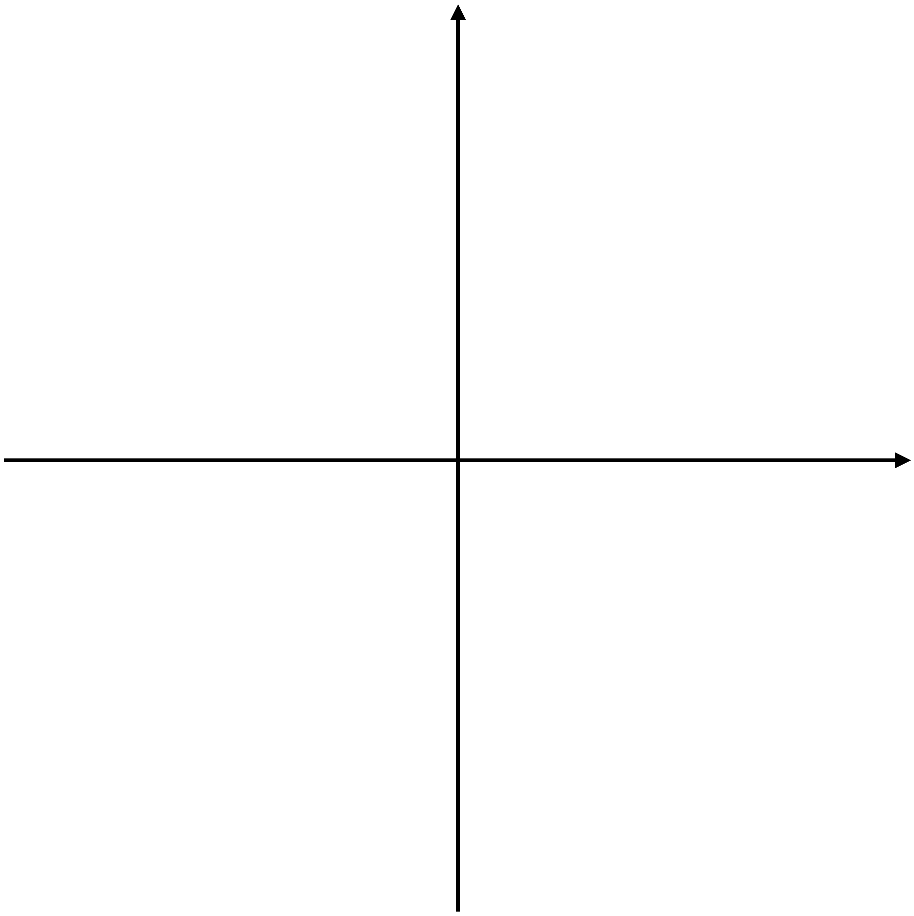
[3 marks]

8 A curve has equation

$$y = \frac{5 - 4x}{1 + x}$$

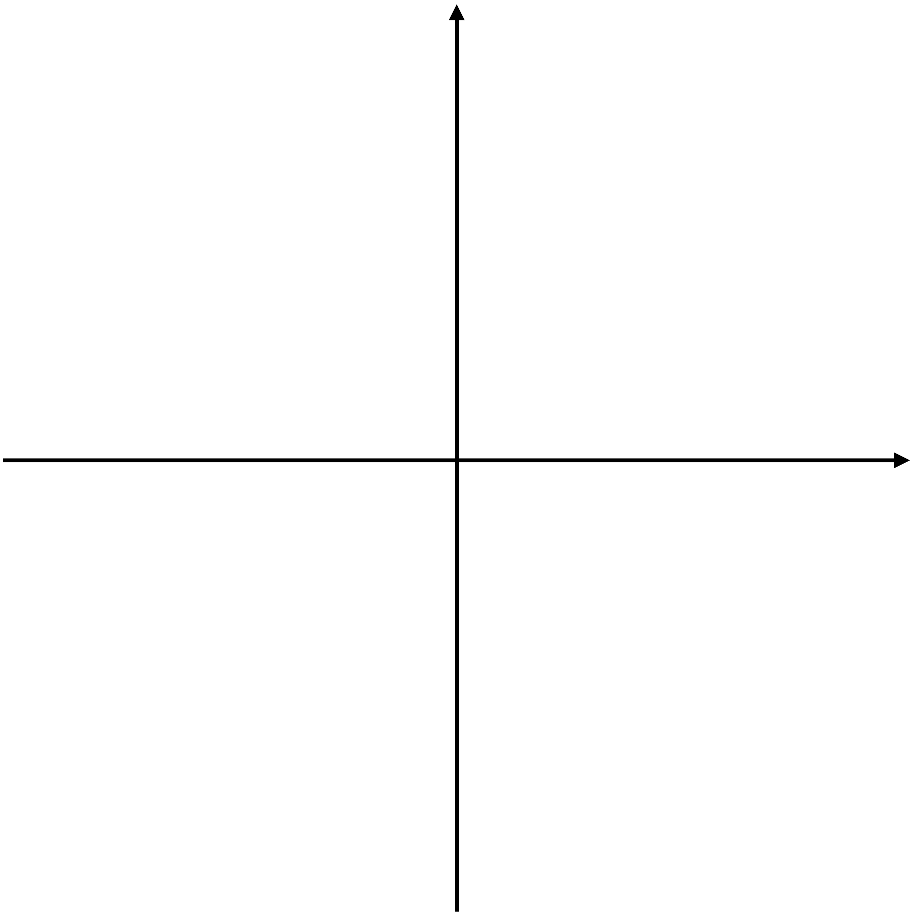
8 (a) Sketch the curve.

[4 marks]



8 (b) Hence sketch the graph of $y = \left| \dfrac{5 - 4x}{1 + x} \right|$.

[1 mark]



9 A line has Cartesian equations $x - p = \dfrac{y+2}{q} = 3 - z$ and a plane has equation $\mathbf{r}.\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = -3$

9 (a) In the case where the plane fully contains the line, find the values of p and q.

[3 marks]

9 (b) In the case where the line intersects the plane at a single point, find the range of values of p and q.

[3 marks]

9 (c) In the case where the angle θ between the line and the plane satisfies $\sin\theta = \frac{1}{\sqrt{6}}$ and the line intersects the plane at $z = 0$

9 (c) (i) Find the value of q.

[4 marks]

9 (c) (ii) Find the value of p.

[3 marks]

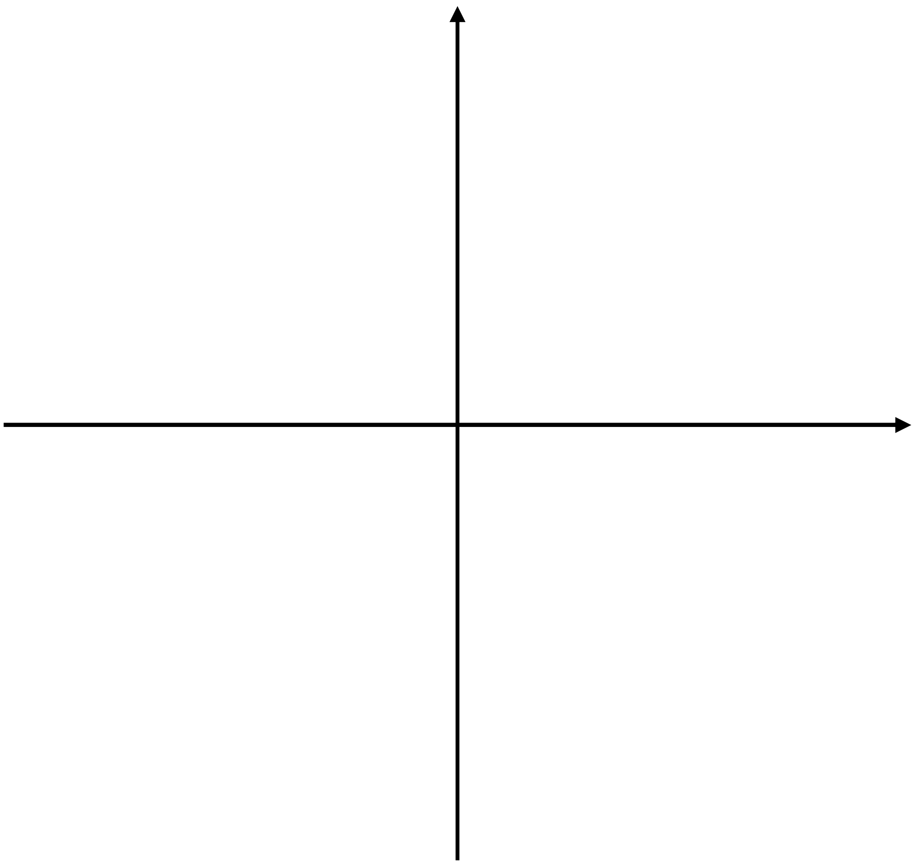
10 The curve, C, has equation $y = \dfrac{x}{\cosh x}$

10 (a) Show that the x-coordinates of any stationary points of C satisfy the equation $\tanh x = \dfrac{1}{x}$

[3 marks]

10 (b) (i) Sketch the graphs of $y = \tanh x$ and $y = \dfrac{1}{x}$ on the axes below.

[2 marks]



10 (b) (ii) Hence determine the number of stationary points of the curve *C*.

[1 mark]

10 (c) Show that $\dfrac{d^2y}{dx^2} + y = 0$ at each of the stationary points of the curve *C*.

[4 marks]

11 (a) Prove that $\frac{\sinh\theta}{1+\cosh\theta}+\frac{1+\cosh\theta}{\sinh\theta}\equiv 2\coth\theta$

Explicitly state any hyperbolic identities that you use within your proof.

[4 marks]

11 (b) Solve $\frac{\sinh\theta}{1+\cosh\theta}+\frac{1+\cosh\theta}{\sinh\theta}=4$ giving your answer in an exact form.

[2 marks]

13 Given that $\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $\mathbf{M}^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ for all $n \in \mathbb{N}$

[5 marks]

Q	Marking Instructions	AO	Marks	Typical Solution
1	Circles correct answer	AO2.2a	B1	$\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$
	Total		**1**	
2	Recalls correct definitions of cosh x and sinh x	AO1.2	B1	$\cosh^2 x - \sinh^2 x \equiv \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$ $\equiv \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$ $\equiv \frac{4}{4}$ $\equiv 1$
	Demonstrates clearly that $\cosh^2 x - \sinh^2 x \equiv 1$ **AG** Award only for completely correct argument including expansion and simplification	AO2.1	R1	
	Total		**2**	

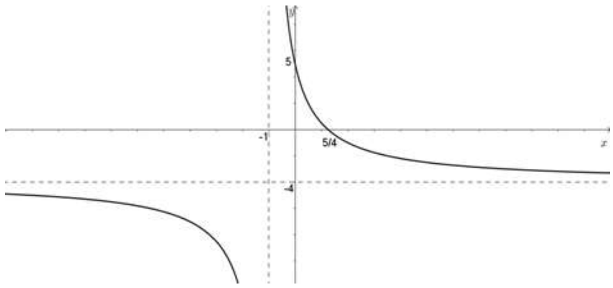
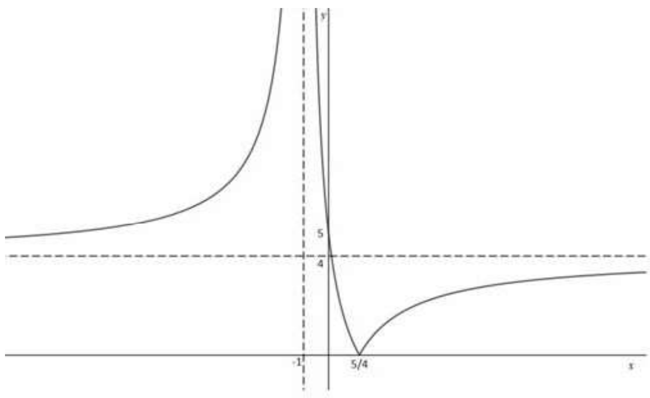
Q	Marking Instructions	AO	Marks	Typical Solution
3(a)	Forms identity using the numerators from each side	AO1.1a	M1	$\frac{2}{(r+1)(r+2)(r+3)} \equiv \frac{A}{(r+1)(r+2)} + \frac{B}{(r+2)(r+3)}$
	Obtains the correct values of A and B	AO1.1b	A1	$\Rightarrow 2 \equiv A(r+3) + B(r+1)$ $\Rightarrow A = 1, B = -1$
(b)	Uses 'their' result from part **(a)** to write fraction as sum of differences. Ignore $\frac{1}{2}$ at this stage.	AO1.1a	M1	$\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2}\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)} - \frac{1}{(r+2)(r+3)}$ $\sum_{r=9}^{97} \frac{1}{(r+1)(r+2)} - \frac{1}{(r+2)(r+3)}$
	Clearly shows step of cancelling of terms	AO2.4	M1	$= \frac{1}{10 \times 11}$ $- \frac{1}{11 \times 12} + \frac{1}{11 \times 12}$ $- \frac{1}{12 \times 13} + \frac{1}{12 \times 13}$
	Obtains correct two term difference Ft 'their' values for A and B provided that 'their' A = − 'their' B	AO1.1b	A1	… $- \frac{1}{98 \times 99} + \frac{1}{98 \times 99}$
	States that method of differences gives $2 \times \sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)}$ so divides their answer by 2 to obtain correct rational solution from fully correct working **AG** (If student merely divides by 2 without justification withhold this mark)	AO2.1	R1	$- \frac{1}{99 \times 100}$ $= \frac{1}{10 \times 11} - \frac{1}{99 \times 100}$ $= \frac{89}{9900}$ $\therefore \sum_{r=9}^{97} \frac{1}{(r+1)(r+2)(r+3)}$ $= \frac{1}{2} \times \frac{89}{9900}$ $= \frac{89}{19800}$
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
5(a)	Makes a correct deduction about another root (PI)	AO2.2a	B1	$(z-(2-3i))(z-(2+3i)) = z^2 - 4z + 13$
	Finds quadratic factor by expanding brackets or using sum and product of roots	AO1.1a	M1	$\therefore$ $\text{p}(z) = (z^2 - 4z + 13)(z^2 + cz + d)$ $(z^2 - 4z + 13)(z^2 + cz + d) \equiv z^4 + 3z^2 + az + b$
	Finds a correct quadratic factor	AO1.1b	A1	
	Compares coefficients with quartic $z^4 + 3z^2 + az + b$	AO1.1a	M1	$c - 4 = 0$ $13 - 4c + d = 3$ $\Rightarrow c = 4,\ d = 6$
	States the correct product of quadratic factors	AO1.1b	A1	$\therefore$ $\text{p}(z) = \left(z^2 - 4z + 13\right)\left(z^2 + 4z + 6\right)$
ALT	Makes a correct deduction about another root	AO2.2a	B1	$(z-(2-3i))(z-(2+3i)) = z^2 - 4z + 13$ $\therefore$ $\text{p}(z) = (z^2 - 4z + 13)(z^2 + cz + d)$
	Finds quadratic factor by expanding brackets or using sum and product of roots	AO1.1a	M1	$(z^2 - 4z + 13)(z^2 + cz + d) \equiv z^4 + 3z^2 + az + b$
	Obtains a correct quadratic factor	AO1.1b	A1	$\alpha + \beta + \gamma + \delta = 0 \ \Rightarrow\ \gamma + \delta = -4$ $\therefore c = 4$
	Uses coefficients/roots to set up equations and find required coefficients	AO1.1a	M1	$\left(\sum \alpha\right)^2 = \sum \alpha^2 + 2\sum \alpha\beta$ $0 = \sum \alpha^2 + 2 \times 3$ $0 = (2-3i)^2 + (2+3i)^2 + \gamma^2 + \delta^2 + 6$
	States the correct product of quadratic factors	AO1.1b	A1	$\therefore \gamma^2 + \delta^2 = 4$ $2\gamma\delta = (\gamma + \delta)^2 - \gamma^2 + \delta^2$ $\gamma\delta = \dfrac{16-4}{2}$ $\therefore d = 6$ $\text{p}(z) = \left(z^2 - 4z + 13\right)\left(z^2 + 4z + 6\right)$

Q	Marking Instructions	AO	Marks	Typical Solution
(b)	States all four correct solutions FT ‘their’ two quadratic factors from part **(a)** provided both M1 marks have been awarded	AO1.1b	B1F	$z = 2 \pm 3\text{i}, -2 \pm \sqrt{2}\text{i}$
	Total		**6**	

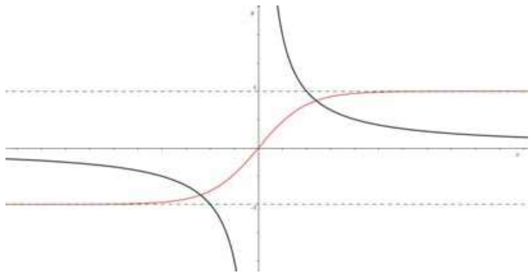
Q	Marking Instructions	AO	Marks	Typical Solution
7(a)	Uses an appropriate method for finding the values of k (for example expanding appropriate determinant)	AO1.1a	M1	$\begin{vmatrix} 1 & -1 & k \\ k & -3 & 5 \\ 1 & -2 & 3 \end{vmatrix} = 0$
	Obtains a quadratic equation in k	AO1.1a	M1	$\begin{vmatrix} -3 & 5 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} k & 5 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} k & -3 \\ 1 & -2 \end{vmatrix} = 0$
	Obtains two correct values for k	AO1.1b	A1	$1 + 3k - 5 + k(-2k + 3) = 0$ $-2k^2 + 6k - 4 = 0$ $k^2 - 3k + 2 = 0$ $(k-2)(k-1) = 0$ $k = 2$ or 1
(b)	Selects an appropriate method to determine the appropriate geometrical configuration and substitutes 'their' first value of k	AO3.1a	M1	when $k = 1$ $x - y + z = 3$ $x - 3y + 5z = -1$ $x - 2y + 3z = -4$
	Eliminates one variable or uses row reduction	AO1.1a	M1	$-2y + 4z = -4$ $y - 2z = 7$
	Obtains a contradiction and makes correct deduction about the geometric configuration (must have correct value for k)	AO2.2a	R1	$y - 2z = 2;\quad y - 2z = 7$ Hence equations are inconsistent and the three planes form a prism
	Substitutes 'their' 2nd value of k into selected method to determine the appropriate geometrical configuration	AO1.1a	M1	when $k = 2$ $x - y + 2z = 3$ $2x - 3y + 5z = -1$ $x - 2y + 3z = -4$
	Obtains a consistent set of equations and makes correct deduction about geometric configuration (must have correct value for k)	AO2.2a	R1	$R_2 - 2R_1 : -y + z = -7$ $R_3 - R_1 : -y + z = -7$ Hence equations are consistent and the three planes form a sheaf – they meet in line

Q	Marking Instructions	AO	Marks	Typical Solution
(c)	Deduces that the planes must meet in a line and hence that $k=2$	AO2.2a	R1	$x-y+2z=3$ $2x-3y+5z=-1$ $x-2y+3z=-4$ $\Rightarrow -y+z=-7$ Let $z=\lambda$ Then $y=\lambda+7$ and $x=3+y-2z$ $=3+\lambda+7-2\lambda$ $=-\lambda+10$
	Selects method to find solution: For example, sets one variable = λ, substitutes and attempts to find other variables in terms of λ	AO1.1a	M1	**ALT** $\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}10\\7\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}$
	Fully states correct solution CAO	AO1.1b	A1	
	Total		**11**	

Q	Marking Instructions	AO	Marks	Typical Solution
8(a)	Indicates correct coordinate or intercept on x and y axes	AO1.1b	B1	$x=0$ $\Rightarrow y=5$ $\Rightarrow (0,5)$ $y=0$ $\Rightarrow 5-4x=0$ $\Rightarrow x=\frac{5}{4}$ $\Rightarrow (\frac{5}{4},0)$
	Indicates correct vertical or horizontal asymptote	AO1.1b	B1	$x=-1$ As $x\to\infty$, $y\to -4$ $y=-4$
	Sketches correct shape of curve	AO1.2	B1	
	Draws fully correct sketch including intercepts and both asymptotes marked	AO1.1b	B1	
(b)	Draws sketch fully correct including shape at x-intercept both asymptotes marked	AO1.1b	B1	
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
9(a)	Uses an appropriate method for ensuring the line lies in the plane	AO3.1a	M1	Let $\lambda = x - p = \dfrac{y+2}{q} = 3 - z$, then $x = \lambda + p,\ y = q\lambda - 2,\ z = 3 - \lambda$
	Obtains equation(s) in p and q	AO1.1a	M1	sub into equation of plane $(\lambda + p) - (q\lambda - 2) - 2(3 - \lambda) = -3$ $\lambda(3-q) + (p-1) = 0$ this is true for all λ therefore p = 1 and q = 3
	Deduces the values of p and q	AO2.2a	A1	ALT vector equation of line is $\mathbf{r} = \begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix}$ therefore $\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix}$ lies on the plane $\begin{pmatrix} p \\ -2 \\ 3 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + 3 = 0$ And $\begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ therefore $\begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$ $\Rightarrow q = 3$ and $p = 1$
(b)	States that to have a solution the coefficient of λ in equation from **(a)** cannot be 0 OR dot product must ≠ 0	AO2.4	R1	$\begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \neq 0 \Rightarrow q \neq 3$
	Deduces the range of values for q	AO2.2a	R1	
	Deduces correct range of values for p	AO2.2a	R1	p can take any value

Q	Marking Instructions	AO	Marks	Typical Solution
(c)(i)	Finds the correct scalar product of the normal to the plane and the direction vector	AO1.1b	B1	$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} 1 \\ q \\ -1 \end{pmatrix}$ $\mathbf{n.d} = 3 - q$
	Correctly deduces the value of cos α	AO2.2a	R1	Let α be angle between the line and the normal to the plane $\sin\theta = \frac{1}{\sqrt{6}} \Rightarrow \cos\alpha = \frac{1}{\sqrt{6}}$
	Forms an equation connecting all relevant parts using $\mathbf{n.d} = \lvert\mathbf{n}\rvert\lvert\mathbf{d}\rvert\cos\theta$	AO3.1a	M1	$q - 3 = \sqrt{6}\sqrt{q^2+2} \times \left(\frac{1}{\sqrt{6}}\right)$
	Obtains correct value for q	AO1.1b	A1	$(3-q)^2 = q^2 + 2$ $\Rightarrow 6q = 7$ giving $q = \frac{7}{6}$
(c)(ii)	Uses 'their' expressions for x and y and 'their' value for q and the equation of the plane to form an equation to find p	AO3.1a	M1	$x - p = \frac{y+2}{\frac{7}{6}} = 3 - z$ $z = 0 \Rightarrow x = p + 3, y = 1.5$
	Uses $z = 0$ to deduce expressions for x and y in terms of p and q	AO2.2a	R1	$\begin{pmatrix} p+3 \\ 1.5 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -3$
	Obtains the correct value of p CAO	AO1.1b	A1	$\Rightarrow p + 3 - 1.5 = -3$ $\Rightarrow p = -4.5$
	Total		**13**	

Q	Marking Instructions	AO	Marks	Typical Solution
10(a)	Uses quotient or product rule to obtain correct derivative	AO1.1b	B1	$y=\dfrac{x}{\cosh x}\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\cosh x-x\sinh x}{\cosh^2 x}$
	Clearly sets 'their' $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ numerator equal to 0	AO2.4	R1	Stationary point $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow\dfrac{\cosh x-x\sinh x}{\cosh^2 x}=0$
	Rearranges to complete a rigorous argument to show the required result. **AG**	AO2.1	R1	$\Rightarrow\cosh x-x\sinh x=0$ $\Rightarrow\dfrac{\sinh x}{\cosh x}=\dfrac{1}{x}$ $\Rightarrow\tanh x=\dfrac{1}{x}$
(b)(i)	Sketches $\tanh x$ correctly including asymptotes	AO1.2	B1	
	Sketches $\dfrac{1}{x}$ correctly	AO1.2	B1	
(ii)	Deduces correct number of stationary points FT 'their' sketch in **(b)(i)**	AO2.2a	B1F	2 stationary points

Q	Marking Instructions	AO	Marks	Typical Solution
10(c)	Finds the second derivative	AO1.1a	M1	$\frac{d^2y}{dx^2}=\frac{\cosh^2 x(\sinh x - x\cosh x - \sinh x)}{\cosh^4 x}$
	Obtains a correct expression for the second derivative	AO1.1b	A1	$-\frac{2\cosh x\sinh x(\cosh x - x\sinh x)}{\cosh^4 x}$
	Deduces that the second term is zero by using results from part **(a)**	AO2.2a	R1	second term is zero at stationary points $\frac{d^2y}{dx^2}=-\frac{x}{\cosh x}=-y$
	Completes a rigorous argument to show the required result. AG Mark awarded if they have a completely correct solution, which is clear, easy to follow and contains no slips	AO2.1	R1	$\Rightarrow \frac{d^2y}{dx^2}+y=0$
	Total		**10**	

Q	Marking Instructions	AO	Marks	Typical Solution
11(a)	Commences proof by considering one side of the identity only: if considering LHS combines terms as a single fraction with a common denominator. If considering RHS writes coth θ as a fraction and introduces factor of (1 + cosh θ to both numerator and denominator) Note alternative valid approaches include commencing proof by considering LHS minus RHS or LHS divided by RHS	AO2.1	R1	$\dfrac{\sinh\theta}{1+\cosh\theta}+\dfrac{1+\cosh\theta}{\sinh\theta}$ $\equiv \dfrac{\sinh^2\theta+1+\cosh^2\theta+2\cosh\theta}{(1+\cosh\theta)\sinh\theta}$ $\equiv \dfrac{\cosh^2\theta+\cosh^2\theta+2\cosh\theta}{(1+\cosh\theta)\sinh\theta}$, $\because 1+\sinh^2\theta\equiv\cosh^2\theta$ $\equiv \dfrac{2\cosh\theta(1+\cosh\theta)}{(1+\cosh\theta)\sinh\theta}$ $\equiv \dfrac{2\cosh\theta}{\sinh\theta}$ $\equiv 2\coth\theta$
	Explicitly states identity $\cosh^2\theta - \sinh^2\theta \equiv 1$ and uses it to eliminate (or introduce) $\sinh^2\theta$	AO2.4	R1	
	Factorises numerator and cancels correctly for 'their' fraction (if considering RHS rearranges 'their' numerator correctly into two factorised expressions)	AO1.1b	B1F	
	Completes rigorous proof to obtain result **AG** Only award if they have a completely correct argument, which is clear and contains no slips.	AO2.1	R1	
(b)	Uses result from part **(a)** to deduce that tanh $\theta = \dfrac{1}{2}$	AO2.2a	R1	$2\coth\theta = 4$ $\tanh\theta = \dfrac{1}{2}$ $\theta = \tanh^{-1}\dfrac{1}{2} = \dfrac{1}{2}\ln 3$
	Uses natural log form and substitutes correct value to obtain correct exact form	AO1.1b	A1	
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
13	Uses proof by induction and investigates formula for n = 1 and $n = k$ (must see evidence of both n = 1 and $n = k$ being considered)	AO3.1a	M1	Using induction method, Let P(n) be the statement $\mathbf{M}^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$
	Demonstrates that formula is true for n = 1	AO1.1b	A1	For n =1 $\begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{M}^1$ ∴P(1) is true
	States assumption that formula true for $n = k$ and uses $\mathbf{M}^{k+1} = \mathbf{M} \times \mathbf{M}^k$	AO2.1	R1	Assume P(k) is true $\mathbf{M}^{k+1} = \mathbf{M} \times \mathbf{M}^k$
	Deduces that formula is also true for $n = k + 1$ from correct working	AO2.2a	R1	$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$ (since P(k) is true)
	Completes a rigorous argument and explains how their argument proves the required result. **AG**	AO2.4	R1	$= \begin{bmatrix} (3^{k-1}+3^{k-1}+3^{k-1}) & (3^{k-1}+..) & (3^{k-1}+.. \\ (3^{k-1}+3^{k-1}+3^{k-1}) & (3^{k-1}+..) & (3^{k-1}+.. \\ (3^{k-1}+3^{k-1}+3^{k-1}) & (3^{k-1}+..) & (3^{k-1}+.. \end{bmatrix}$ But $3^{k-1} + 3^{k-1} + 3^{k-1} = 3 \times 3^{k-1}$ $= 3^k$ Hence $\mathbf{M}^{k+1} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$ $\therefore \mathbf{M}^{k+1} = \begin{bmatrix} 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \end{bmatrix}$ ∴P(k +1) is true Since P(1) is true *and* P(k)⇒P(k + 1) , hence, by induction, P(n) is true for all $n \in \mathbb{N}$
	Total		**5**	

Answer **all** questions in the spaces provided.

1 Given that $z_1 = 4e^{i\frac{\pi}{3}}$ and $z_2 = 2e^{i\frac{\pi}{4}}$

state the value of $\arg\left(\frac{z_1}{z_2}\right)$

Circle your answer.

[1 mark]

$\frac{\pi}{12}$ $\frac{4}{3}$ $\frac{7\pi}{12}$ 2

2 Given that z is a complex number and that z^* is the complex conjugate of z

prove that $zz^* - |z|^2 = 0$

[3 marks]

3 The transformation T is defined by the matrix **M**. The transformation S is defined by the matrix $\mathbf{M}^{-1}$. Given that the point (x, y) is invariant under transformation T, prove that (x, y) is also an invariant point under transformation S.

[3 marks]

4 Solve the equation $z^3 = \mathrm{i}$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$

[4 marks]

5 Find the smallest value θ of for which

$$(\cos\theta + \mathrm{i}\sin\theta)^5 = \frac{1}{\sqrt{2}}(1-\mathrm{i})\ \{\theta \in \mathbb{R} : \theta > 0\}$$

[4 marks]

6 Prove that $8^n - 7n + 6$ is divisible by 7 for all integers $n \geq 0$

[5 marks]

9 A student claims:

"Given any two non-zero square matrices, **A** and **B**, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$"

9 (a) Explain why the student's claim is incorrect giving a counter example.

[2 marks]

9 (b) Refine the student's claim to make it fully correct.

[1 mark]

9 (c) Prove that your answer to part **(b)** is correct.

[3 marks]

12 $\mathbf{M} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$

12 (a) Given that 4 is an eigenvalue of **M**, find a corresponding eigenvector.

[3 marks]

12 (b) Given that **MU** = **UD**, where **D** is a diagonal matrix, find possible matrices for **D** and **U**.

[8 marks]

13 **S** is a singular matrix such that

$$\det \mathbf{S} = \begin{vmatrix} a & a & x \\ x-b & a-b & x+1 \\ x^2 & a^2 & ax \end{vmatrix}$$

Express the possible values of x in terms of a and b.

[7 marks]

14 Given that the vectors **a** and **b** are perpendicular, prove that

$|(\mathbf{a} + 5\mathbf{b}) \times (\mathbf{a} - 4\mathbf{b})| = k|\mathbf{a}||\mathbf{b}|$, where k is an integer to be found.

Explicitly state any properties of the vector product that you use within your proof.

[9 marks]

15 (a) Show that $(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta}) = \frac{1}{16}(17-8\cos 2\theta)$

[3 marks]

15 (b) Given that the series $e^{2i\theta} + \frac{1}{4}e^{4i\theta} + \frac{1}{16}e^{6i\theta} + \frac{1}{64}e^{8i\theta} + ...$ has a sum to infinity, express this sum to infinity in terms of $e^{2i\theta}$

[2 marks]

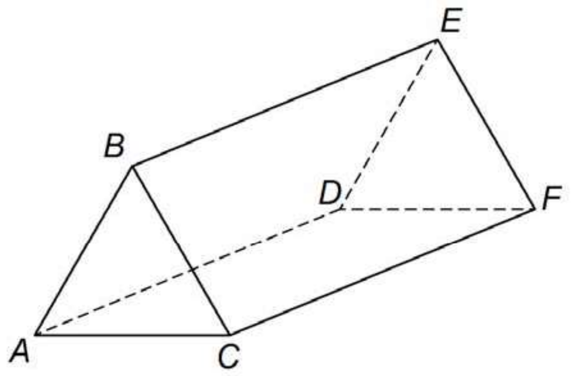
15 (c) Hence show that $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \cos 2n\theta = \frac{16\cos 2\theta - 4}{17 - 8\cos 2\theta}$

[4 marks]

15 (d) Deduce a similar expression for $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \sin 2n\theta$

[1 mark]

16 A designer is using a computer aided design system to design part of a building. He models part of a roof as a triangular prism *ABCDEF* with parallel triangular ends *ABC* and *DEF*, and a rectangular base *ACFD*. He uses the metre as the unit of length.



The coordinates of *B*, *C* and *D* are (3, 1, 11), (9, 3, 4) and (–4, 12, 4) respectively.

He uses the equation $x - 3y = 0$ for the plane *ABC*.

He uses $\left[\mathbf{r} - \begin{pmatrix} -4 \\ 12 \\ 4 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for the equation of the line *AD*.

Find the volume of the space enclosed inside this section of the roof.

[9 marks]

Q	Marking Instructions	AO	Marks	Typical Solution
1	Circles correct answer	AO1.1b	B1	$\frac{\pi}{12}$
	Total		**1**	
2	Defines generalised z and z^* in Cartesian or polar form	AO1.2	B1	Let $z = a + b\text{i}$ then $z^* = a - b\text{i}$
	Expands and simplifies zz^* and $\lvert z\rvert^2$ (at least one correct)	AO1.1b	M1	$zz^* - \lvert z\rvert^2 = (a+b\text{i})(a-b\text{i}) - \left(\sqrt{a^2+b^2}\right)^2$ $= a^2 + ab\text{i} - ab\text{i} - (b\text{i})^2 - (a^2+b^2)$ $= a^2 + b^2 - (a^2+b^2)$ $= 0$
	Completes a well-structured argument to prove the required result. **AG** Mark awarded if they have a completely correct solution, which is clear, easy to follow and contains no slips	AO2.1	R1	ALT Let $z = r\text{e}^{\text{i}\theta}$ then $z^* = r\text{e}^{-\text{i}\theta}$ $zz^* - \lvert z\rvert^2 = r\text{e}^{\text{i}\theta} r\text{e}^{-\text{i}\theta} - r^2$ $= r^2\text{e}^{\text{i}\theta - \text{i}\theta} - r^2$ $= r^2 - r^2$ $= 0$
	Total		**3**	

Q	Marking Instructions	AO	Marks	Typical Solution
3	Commences a proof by correctly setting up an equation using the definition of an invariant point	AO2.1	R1	For an invariant point $\mathbf{M}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$
	Pre-multiplies by $\mathbf{M}^{-1}$.	AO2.1	R1	Pre-multiply both sides by $\mathbf{M}^{-1}$ $\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}x\\y\end{pmatrix}$
	Uses $\mathbf{M}^{-1}\mathbf{M} = \boldsymbol{I}$ and concludes their rigorous mathematical argument to deduce that (x, y) is invariant under S **AG**	AO2.2a	R1	$\mathbf{M}^{-1}\mathbf{M} = \boldsymbol{I}$ hence $\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}x\\y\end{pmatrix}$ Therefore $\begin{pmatrix}x\\y\end{pmatrix}$ is invariant under S.
	Total		**3**	

Q	Marking Instructions	AO	Marks	Typical Solution
4	Expresses i or z in polar form	AO1.2	B1	$\mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$
	Uses de Moivre's Theorem	AO3.1a	M1	$z = \left[\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+2n\pi)}\right]^{\frac{1}{3}} = \left[\mathrm{e}^{\mathrm{i}(\frac{\pi}{6}+\frac{2n\pi}{3})}\right]$
	Finds three consecutive values for θ	AO1.1a	A1	$\frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}\left(\text{or } \frac{3\pi}{2} \text{ etc}\right)$
	Finds all three correct solutions for z	AO1.1b	A1	$z = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ ALT $z = \mathrm{e}^{\mathrm{i}\theta} \Rightarrow z = \cos\theta + \mathrm{i}\sin\theta$ $z^3 = \mathrm{i} \Rightarrow \cos 3\theta + \mathrm{i}\sin 3\theta = \mathrm{i}$ $\therefore \cos 3\theta = 0$ and $\sin 3\theta = 1$ $\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}\left(\text{or } \frac{3\pi}{2} \text{ etc}\right)$ $z = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical Solution
5	Uses de Moivre's theorem	AO3.1a	M1	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \frac{1}{\sqrt{2}}(1-\mathrm{i})$ $\Rightarrow \cos 5\theta + \mathrm{i}\sin 5\theta = \frac{1}{\sqrt{2}}(1-\mathrm{i})$
	Equates real and imaginary parts and obtains two equations	AO1.1a	A1	$\cos 5\theta = \frac{1}{\sqrt{2}} \quad \sin 5\theta = -\frac{1}{\sqrt{2}}$
	Deduces that the smallest possible value of 5θ is $\frac{7\pi}{4}$ FT from 'their' equations provided M1 has been awarded	AO2.2a	A1F	$(5\theta =)\frac{7\pi}{4}$
	Obtains the smallest possible value of θ from fully correct reasoning FT from 'their' 5θ provided M1 has been awarded	AO1.1b	A1F	$\theta = \frac{7\pi}{20}$
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical Solution
6	Uses proof by induction and investigates the expression for n = 0 and n = k (must see evidence of both n = 0 and n = k being considered)	AO3.1a	B1	Let $\mathrm{f}(n)=8^n-7n+6$ $\mathrm{f}(0)=1+6=7$ $\Rightarrow \mathrm{f}(n)$ is divisible by 7 when n = 0 Consider $n = k$ Assume that f(k) is divisible by 7 $\mathrm{f}(k+1)=8^{k+1}-7(k+1)+6$ $\mathrm{f}(k+1)-8\mathrm{f}(k)=56k-7(k+1)+6-48$ $\mathrm{f}(k+1)-8\mathrm{f}(k)=49k-49$ $\mathrm{f}(k+1)=8\mathrm{f}(k)+49(k-1)$ $=8\mathrm{f}(k)+7(7k-7)$ $\therefore \mathrm{f}(k+1)$ is divisible by 7 since $\mathrm{f}(k)$ is divisible by 7 Therefore f(k) is divisible by 7 $\Rightarrow$ f(k + 1) is divisible by 7
	Shows that statement is true for n = 0	AO1.1b	B1	
	Commences argument by considering f(k + 1) in terms of f(k)	AO2.1	R1	
	Makes correct deduction that if f(n) is divisible by 7 then f(n + 1) is also divisible by 7	AO2.2a	R1	
	Completes a rigorous argument and explains how their argument proves the required result. **AG**	AO2.4	R1	Since $\mathrm{f}(0)$ is divisible by 7 *and* f(k) is divisible by 7 $\Rightarrow$f(k + 1) is divisible by 7 then, by induction, $\mathrm{f}(n)=8^n-7n+6$ is divisible by 7 for all integers $n\geq 0$
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
6 ALT				Let $\text{f}(n) = 8^n - 7n + 6$ $\text{f}(0) = 1 + 6 = 7$ $\Rightarrow \text{f}(n)$ is divisible by 7 when n=0 Consider $n = k$ Assume that $\text{f}(k)$ is divisible by 7 $\text{f}(k+1) = 8^{k+1} - 7(k+1) + 6$ $= 8\left(8^k - 7k + 6\right) + 8 \times 7k - 7k - 1 - 48$ $= 8\text{f}(k) + 49k - 49$ $= 8\text{f}(k) + 7(7k - 7)$ $\therefore \text{f}(k+1)$ is divisible by 7 since $\text{f}(k)$ is divisible by 7 Therefore $\text{f}(k)$ is divisible by 7 $\Rightarrow \text{f}(k+1)$ is divisible by 7 Since $\text{f}(0)$ is divisible by 7 *and* $\text{f}(k)$ is divisible by 7 $\Rightarrow \text{f}(k+1)$ is divisible by 7 then, by induction, $\text{f}(n) = 8^n - 7n + 6$ is divisible by 7 for all integers $n \geq 0$
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
9(a)	Explains that the claim is incorrect as **singular** square matrices do not have inverses.	AO2.3	E1	Statement is incorrect if either matrix is singular/has determinant equal to zero as the inverse will not exist
	Correctly gives an example of a singular matrix.	AO1.1b	B1	Eg $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ is singular
(b)	Correctly refines the statement using 'non-singular' or equivalent wording	AO2.3	B1	Given any two **non-singular** square matrices, **A** and **B**, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
(c)	Correctly recalls the inverse property for matrices **A** and **B** (seen at least once)	AO1.2	B1	**A** and **B** are non-singular so inverses exist hence **A** and **B** are non-singular so inverses exist hence
	Correctly uses associativity by regrouping (seen at least once)	AO2.5	B1	$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1}$ $= \mathbf{A}\boldsymbol{I}\mathbf{A}^{-1}$ $= \mathbf{AA}^{-1}$
	Correctly applies the identity property throughout and concludes their rigorous mathematical argument with no errors or omissions	AO2.1	R1	$= \boldsymbol{I}$ Since $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \boldsymbol{I}$ Then $(\mathbf{AB})^{-1} = (\mathbf{B}^{-1}\mathbf{A}^{-1})$
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
12(a)	Forms appropriate equation using **M**λ =λ**v**	AO1.1a	M1	$\begin{bmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 4\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
	Eliminates one variable	AO1.1a	M1	$-5a+2b-2c=0$ $2a-2b-2c=0$
	Deduces a correct eigenvector	AO2.2a	A1	$-a-2b-5c=0$ $3a+3c=0$ eigenvector is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Q	Marking Instructions	AO	Marks	Typical Solution
(b)	Forms the characteristic equation of **M**	AO3.1a	M1	$\begin{vmatrix} -1-\lambda & 2 & -1 \\ 2 & 2-\lambda & -2 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$
	Obtains the correct characteristic equation - unsimplified	AO1.1b	A1	$(-1-\lambda)\left[(2-\lambda)(-1-\lambda)-4\right]$ $-2(-2-2\lambda-2)-1(-4+2-\lambda)=0$
	Obtains roots and identifies them as eigenvalues for 'their' characteristic equation	AO1.1b	A1F	$-\lambda^3+12\lambda+16=0$ $(4-\lambda)(\lambda^2+4\lambda+4)=0$ $-(\lambda+2)(\lambda-4)(\lambda+2)=0$ Eigenvalues are 4, –2, –2
	Forms an appropriate matrix equation using the eigenvalue –2 FT 'their' eigenvalue	AO3.1a	M1	$\begin{bmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
	Expands and simplifies to obtain a single equation in x, y and z FT 'their' matrix equation provided both M1 marks have been awarded	AO1.1b	A1F	$x + 2y - z = 0$
	Correctly deduces two linearly independent eigenvectors CAO	AO2.2a	A1	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
	Correctly identifies that the matrix **D** must include 4 and 'their' other eigenvalue(s)	AO1.2	B1F	$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$
	Correctly identifies the corresponding **U** matrix from 'their' eigenvectors	AO1.1b	A1F	$\mathbf{U} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$
	Total		**11**	

Q	Marking Instructions	AO	Marks	Typical Solution
13	Explains that det**M**=0 when **M** is singular (Seen anywhere)	AO2.4	R1	**S** is singular $\Rightarrow \begin{vmatrix} a & a & x \\ x-b & a-b & x+1 \\ x^2 & a^2 & ax \end{vmatrix} = 0$
	Seeks factor by combining rows or columns to find a first linear factor for example $C_1' = C_1 - C_2$	AO3.1a	M1	det **S** $= \begin{vmatrix} 0 & a & x \\ x-a & a-b & x+1 \\ x^2-a^2 & a^2 & ax \end{vmatrix}$
	Extracts first factor correctly	AO1.1b	A1	$= (x-a)\begin{vmatrix} 0 & a & x \\ 1 & a-b & x+1 \\ x+a & a^2 & ax \end{vmatrix}$
	Combines rows or columns to find a second linear factor $R_3' = R_3 - aR_1$	AO1.1a	M1	det **S** $= (x-a)\begin{vmatrix} 0 & a & x \\ 1 & a-b & x+1 \\ x+a & 0 & 0 \end{vmatrix}$
	Extracts second factor correctly	AO1.1b	A1	$= (x-a)(x+a)\begin{vmatrix} a & x \\ a-b & x+1 \end{vmatrix}$
	Completes expansion and obtains final factor	AO1.1b	A1	$= (x-a)(x+a)(a+bx)$
	Deduces correct values of x FT 'their' factors	AO2.2a	A1F	$(x-a)(x+a)(a+bx) = 0$ $x = a, -a, -\dfrac{a}{b}$
	Total		**7**	

Q	Marking Instructions	AO	Marks	Typical Solution
14	Uses vector product and expands brackets correctly	AO1.1a	M1	$\lvert(\mathbf{a}+5\mathbf{b})\times(\mathbf{a}-4\mathbf{b})\rvert$ $=\lvert\mathbf{a}\times\mathbf{a}-4\mathbf{a}\times\mathbf{b}+5\mathbf{b}\times\mathbf{a}-20\mathbf{b}\times\mathbf{b}\rvert$ $=\lvert\mathbf{0}-4\mathbf{a}\times\mathbf{b}+5\mathbf{b}\times\mathbf{a}-\mathbf{0}\rvert$ since **a** is parallel to **a** and **b** is parallel to **b** then $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ and $\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $=\lvert-4\mathbf{a}\times\mathbf{b}-5\mathbf{a}\times\mathbf{b}\rvert$ since $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ $=\lvert-9\mathbf{a}\times\mathbf{b}\rvert$ $=9\lvert\mathbf{a}\times\mathbf{b}\rvert$ $=9\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 90$ $=9\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$
	Uses the correct notation and correct order with the vector product.	AO2.5	B1	
	Reduces the number of terms in 'their' expression by using $\mathbf{a}\times\mathbf{a}=\mathbf{b}\times\mathbf{b}=\mathbf{0}$	AO1.1a	M1	
	and explains their reasoning (must have clear statement that $\mathbf{a}\times\mathbf{a}=\mathbf{0}$)	AO2.4	E1	
	Uses $-\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$ to collect 'their' terms together	AO1.1a	M1	
	and explains their reasoning (must have clear statement that $-\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$ **OE)**	AO2.4	E1	
	Recalls correctly the formula for the modulus of the vector product (may see $\lvert\mathbf{a}\rvert\times\lvert\mathbf{b}\rvert\sin\theta$ or may see $\lvert\mathbf{a}\rvert\times\lvert\mathbf{b}\rvert\sin 90^\circ$)	AO1.2	B1	
	Obtains $\lvert\mathbf{a}\times\mathbf{b}\rvert=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ since vectors **a** and **b** are perpendicular	AO1.1b	A1	
	Completes a fully correct proof giving an answer of $9\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ CAO	AO2.2a	R1	
	Total		**9**	

Q	Marking Instructions	AO	Marks	Typical Solution
15(a)	Commences an argument by correctly expanding brackets and simplifying final term to 1/16	AO1.1a	M1	$(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})$ $=1-\frac{1}{4}e^{2i\theta}-\frac{1}{4}e^{-2i\theta}+\frac{1}{16}$ $=\frac{17}{16}-\frac{1}{4}(\cos 2\theta+i\sin 2\theta)-\frac{1}{4}(\cos 2\theta-i\sin 2\theta)$ $=\frac{17}{16}-\frac{1}{2}\cos 2\theta$ $=\frac{1}{16}(17-8\cos 2\theta)$
	Substitutes correctly for both $e^{2i\theta}$ and $e^{-2i\theta}$ in terms of $\cos 2\theta$ and $\sin 2\theta$ (seen anywhere in solution)	AO1.1b	B1	
	Completes argument and reaches stated result by collecting terms and simplifying correctly, no errors in working seen **AG**	AO2.1	R1	
(b)	Identifies series as a geometric series and states first term and common ratio correctly	AO1.1b	B1	Geometric series with first term $r=e^{2i\theta}$ and common ratio $a=\frac{1}{4}e^{2i\theta}$
	States and uses sum to infinity formula correctly FT incorrect values for first term and common ratio	AO1.1b	B1F	$S_\infty=\frac{a}{1-r}=\frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}$

Q	Marking Instructions	AO	Marks	Typical Solution
(c)	Deduces that the series in part **(c)** is related to the real part of the series in part **(b)**	AO2.2a	R1	Series stated = real part of the series $e^{2i\theta}+\frac{1}{4}e^{4i\theta}+\frac{1}{16}e^{6i\theta}+\frac{1}{64}e^{8i\theta}+....$
	Selects an appropriate method by using the result in part **(b)** and multiplying appropriately to realise the denominator	AO3.1a	M1	Using result from previous part $\frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}=\frac{e^{2i\theta}}{(1-\frac{1}{4}e^{2i\theta})}\times\frac{(1-\frac{1}{4}e^{-2i\theta})}{(1-\frac{1}{4}e^{-2i\theta})}$
	Substitutes to obtain an expression with cosines and sines only – using part **(a)** FT incorrect sum to infinity provided M1 has been awarded	AO1.1b	A1F	$=\frac{e^{2i\theta}-\frac{1}{4}}{(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})}$ $\frac{\cos 2\theta-\frac{1}{4}+i\sin 2\theta}{\frac{1}{16}(17-8\cos 2\theta)}$
	Identifies the real part and correctly completes the argument to reach the stated result. Only award for an error-free fully correct solution	AO2.1	R1	Real part = $\frac{\cos 2\theta-\frac{1}{4}}{\frac{1}{16}(17-8\cos 2\theta)}=\frac{16\cos 2\theta-4}{17-8\cos 2\theta}$
(d)	Identifies the imaginary part and states the correct expression	AO2.2a	R1	Required series = imaginary part of the given series hence $\frac{\sin 2\theta}{\frac{1}{16}(17-8\cos 2\theta)}=\frac{16\sin 2\theta}{17-8\cos 2\theta}$
	Total		**10**	

Q	Marking Instructions	AO	Marks	Typical Solution
16	Uses the mathematical model to find the volume by first finding the coordinate of *A*. To award this mark must see an attempt to find coords of *A*, **and** an attempt at volume of prism	AO3.4	M1	$x = 4t - 4$ $y = 12 - 12t$ $z = 4$ $4t - 4 - 3(12 - 12t) = 0$
	Selects method involving both equation of plane and equation of line to find coords of *A* **Either** using parametric form **or** using cross product Ignore sign errors	AO3.1a	M1	$40t$-$40 = 0$ $t = 1$ $\begin{pmatrix} 0 & 0 & 4 \end{pmatrix}$ OR
	Either collects terms together and solves to find value of parameter for 'their' equation **Or** correctly calculates cross product for 'their' vectors	AO1.1b	A1F	$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -4 \\ 12 \\ 4 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
	Deduces the correct coordinates of *A*	AO2.2a	A1	$\begin{pmatrix} 3y+4 \\ y-12 \\ z-4 \end{pmatrix} \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $12(z-4) = 0 \Rightarrow z = 4$
	Selects a correct approach to calculate the volume of the prism.	AO3.1a	M1	$-12(3y+4) - 4(y-12) = 0$ $\Rightarrow y = 0, x = 0$ *A* has coordinates $(0,0,4)$
	Finds two sides of the triangle *ABC* in vector form FT 'their' *A*	AO1.2	A1F	$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$
	Finds area of *ABC* FT 'their' *A*	AO1.1b	A1F	$\overrightarrow{AC} = \begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix}$
	Finds length of prism FT 'their' *A*	AO1.1b	A1F	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -21 \\ 63 \\ 0 \end{pmatrix}$
	Gives their answer in context by correctly finding the volume of the roof with correct units. FT 'their' prism	AO1.1b	A1F	Area *ABC* $= \dfrac{21\sqrt{10}}{2}$ $d = 4\sqrt{10}$ Volume = $V = \dfrac{21\sqrt{10}}{2} \times 4\sqrt{10} = 420\text{ m}^3$
	Total		**9**	
	Total		**100**	